

Density growth in Kantowski-Sachs cosmologies with cosmological constant

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Abstract. In this work the growth of density perturbations in Kantowski-Sachs cosmologies with a positive cosmological constant is studied, using the 1+3 and 1+1+2 covariant formalisms. For each wave number we obtain a closed system for scalars formed from quantities that are zero on the background and hence are gauge-invariant. The solutions to this system are then analyzed both analytically and numerically. In particular the effects of anisotropy and the behaviour close to a bounce in the cosmic scale factor are considered. We find that typically the density gradient in the bouncing direction experiences a local maximum at or slightly after the bounce.

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1. Introduction

The observed distribution of inhomogenities and anisotropies in the background radiation seems to be well described by the Λ CDM model, see e.g. [1, 2]. However they do not match perfectly with this model, [3, 4, 5], and consequently it is of interest to explore the complete phase-space of cosmological models and to see how differences from the standard picture affect the basic properties of the universe.

Perturbations of anisotropic cosmological models have been considered by many authors, e.g., [6, 7, 8, 9, 10], using methods depending on the choice of gauge, like the perturbation theory of Lifshitz and Khalatnikov [11] or Bardeen's gauge invariant theory [12]. Unfortunately, the variables in Bardeen's theory are defined with respect to a particular coordinate system, making their geometrical and physical meaning unclear [13]. In the covariant approaches one circumvents these problems by using the spatial curvature rather than the metric as defining variables [14, 15] and as the perturbed variables choosing objects that are zero on the background [16, 17], and hence are gauge invariant [18]. Some works on perturbations in anisotropic cosmological models along these lines can be found in [19, 20].

Bouncing cosmologies are of interest since the initial singularity can be avoided in these models. The observable universe seems to be close to homogeneous and isotropic, see e.g. [21]. Hence it may be approximated by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric that also serves as isotropic limits of different Bianchi models. In particular, the flat, open and closed FLRW universes are isotropic limits of Bianchi I, V and IX models, respectively [22]. From the Raychaudhuri equation it follows that a bounce is not possible in FLRW models if the strong energy condition is required to hold [23]. Nevertheless, when the weak energy condition holds a bounce can occur in a closed FLRW universe [24], [25], while matter violating the null energy condition can cause a bounce in flat or open universes [26]-[29]. Bouncing FLRW universes are also suggested by alternative theories of gravitation such as $f(R)$ gravity [30], $f(T)$ gravity [31] and braneworlds [32], or by loop quantum gravity [33]-[36].

Cosmic microwave background (CMB) observations support a close to flat FLRW ($|\Omega_k| < 10^{-2}$) universe [21]. Such constraints from the CMB observations have also been considered in bouncing cosmologies. Some models wherein a bounce appears due to a scalar field suffer from high tensor-to-scalar ratio [27], [37]. However, this can be remedied both in general relativity [38] and $f(T)$ gravity [31] by introducing another massless scalar field. It was shown in loop quantum cosmology for a bounce followed by an inflationary phase that the tensor-to-scalar ratio places within the observational bound and that the low CMB multipoles are suppressed [39]-[41]. Actually, this suppression is observed but it can also have other origins [42]. Bouncing FLRW models with dust, radiation and cosmological constant are excluded by observations due to an elegant argument by Börner and Ehlers [68]. If the bounce took place before the formation of the quasars, that are observed at larger redshifts than 4, the present fraction of matter relative to the critical one would be less than 0.02, in contradiction with current estimates.

It is not only the FLRW cosmologies that are interesting as possible models of the universe. The other models that have been investigated in some depth are those that have homogeneous but anisotropic geometries, namely the Bianchi and Kantowski-Sachs [43] cosmologies. Bianchi models were also used successfully to explain the CMB spectrum [44]-[47] and Kantowski-Sachs cosmologies might also be relevant for this purpose. The particular model presented in [48] can explain some features of the

CMB spectrum, however not all.

The cosmic no-hair conjecture says that spacetimes containing a positive cosmological constant finally evolves into de Sitter state. Wald proved that this happens for all Bianchi models with exception of Bianchi IX [49]. It was assumed in the proof that the fluid congruences are orthogonal to the homogeneous symmetry surfaces and that the matter energy-momentum tensor, not including the cosmological constant, satisfies the dominant and strong energy conditions. The isotropisation also happens under inflation provided via a scalar field with an exponential potential [50]-[52]. However, in the tilted case isotropisation does not necessarily take place even during inflation [24], [53]. For Bianchi IX universes Wald gave a sufficient condition for them to evolve into de Sitter state. The evolution of Kantowski-Sachs universes with positive cosmological constant show similarities with that of Bianchi IX cosmologies [54]. Although not the all initial conditions lead to de Sitter state, Moniz showed that the cosmic no-hair conjecture is widely valid in Kantowski-Sachs universes. Basically, the Kantowski-Sachs cosmologies with positive cosmological constant can evolve into de Sitter or Kasner states [24].

In this paper we investigate the density growth in perfect fluid Kantowski-Sachs cosmologies with positive cosmological constant mainly in those cases when isotropisation happens. We consider both bouncing and non-bouncing types of evolutions. The conditions under which bounces are possible were studied in [55] and with a positive cosmological constant the Kantowski-Sachs models may under certain conditions undergo a bounce. This effect can also be achieved for Kantowski-Sachs solutions in R^n gravity, see [56]. To study density perturbations we use the 1+3 and 1+1+2 covariant splits of spacetime [16, 57, 58, 59]. As inhomogeneity variables the spatial gradients of the density, the expansion, the shear scalar and one more auxiliary scalar to close the system, are used. These quantities are zero on the background, and are hence gauge invariant. By projecting along the preferred and orthogonal directions respectively, taking divergences of these projections and making harmonic decompositions of the spatial derivatives, the system is reduced to a first order system in time of four scalar quantities for each wave number.

In section 2 we briefly review the 1+3 and 1+1+2 covariant splits of spacetime. Then in section 3 the background solutions, of Kantowski-Sachs type, are studied and all background vacuum solutions are given. The perturbative equations are determined in section 4 and some analytical results are obtained in 5. Numerical studies of perturbations on different backgrounds, both with and without bounces, are performed in section 6. In Appendix E some results on bouncing closed FLRW universes are given for reference.

2. The 1+3 and 1+1+2 covariant formalisms

The 1+3 and 1+1+2 covariant splits of spacetime are here briefly reviewed. These formalisms are suitable also for perturbative calculations, as will be done in section 4. For more details see, e.g., [16, 57] and [58, 59].

2.1. Preliminaries

In [16, 57] a covariant formalism for the 1+3 split of spacetimes with a preferred time-like vector, u^a , was developed. The projection operator onto the perpendicular

3-space is given by

$$h_a^b = g_a^b + u_a u^b. \quad (1)$$

Projections with h_a^b of vectors are denoted by angle brackets $\psi^{<a>} \equiv h_b^a \psi^b$ and the projected symmetric trace-free (PSTF) of a tensor is given by

$$\psi^{<ab>} \equiv \left[h_c^{(a} h_d^{b)} - \frac{1}{3} h^{ab} h_{cd} \right] \psi^{cd}. \quad (2)$$

The covariant time derivative and projected derivative are given by

$$\dot{\psi}_{a...b} \equiv u^c \nabla_c \psi_{a...b} \quad (3)$$

and

$$\tilde{\nabla}_c \psi_{a...b} \equiv h_c^f h_a^d \dots h_b^e \nabla_f \psi_{d...e} \quad (4)$$

respectively. The covariant derivative of the 4-velocity, u^a , can be decomposed as

$$\nabla_a u_b = -u_a \dot{u}_b + \tilde{\nabla}_a u_b = -u_a \dot{u}_b + \frac{1}{3} \theta h_{ab} + \omega_{ab} + \sigma_{ab} \quad (5)$$

where $\dot{u}_a \equiv u^b \nabla_b u_a$ is the acceleration, $\theta \equiv \tilde{\nabla}_a u^a$ the expansion, $\sigma_{ab} \equiv \tilde{\nabla}_{<a} u_{b>}$ the shear and $\omega_{ab} \equiv \tilde{\nabla}_{[a} u_{b]}$ the vorticity of u^a .

A formalism for a further split (1+2) with respect to a spatial vector n^a (with $u^a n_a = 0$) was then developed in [58, 59]. Projections perpendicular to n^a are made with

$$N_a^b = g_a^b + u_a u^b - n_a n^b. \quad (6)$$

Projected vectors $v^{<a>}$ can be decomposed with respect to n^a as

$$v^{<a>} = V n^a + \bar{V}^a \quad (7)$$

with

$$V \equiv n^a v_a \quad \text{and} \quad \bar{V}^a \equiv N^{ab} v_b \equiv v^{\bar{a}}, \quad (8)$$

where a bar over an index denotes projection with N^{ab} . Similarly PSTF tensors $\psi_{<ab>}$ can be decomposed as

$$\psi_{<ab>} = \Psi \left(n_a n_b - \frac{1}{2} N_{ab} \right) + 2 \Psi_{(a} n_{b)} + \Psi_{ab}, \quad (9)$$

where

$$\Psi \equiv n^a n^b \psi_{<ab>} \quad (10)$$

$$\Psi_a \equiv N_a^b n^c \psi_{<bc>} \quad (11)$$

$$\Psi^{ab} \equiv \left[N_c^{(a} N_d^{b)} - \frac{1}{2} N^{ab} N_{cd} \right] \psi^{<cd>} \equiv \Psi^{\{ab\}}. \quad (12)$$

Here $\Psi^{\{ab\}}$ is symmetric and traceless.

Derivatives along and perpendicular to n^a are given by

$$\hat{\psi}_{a...b} \equiv n^c \tilde{\nabla}_c \psi_{a...b} = n^c h_c^f h_a^d \dots h_b^e \nabla_f \psi_{d...e} \quad (13)$$

and

$$\delta_c \psi_{a...b} \equiv N_c^f N_a^d \dots N_b^e \tilde{\nabla}_f \psi_{d...e} \quad (14)$$

respectively. Similarly to the derivative of the 4-velocity, the derivatives of n_a can be decomposed as

$$\tilde{\nabla}_a n_b = n_a a_b + \frac{1}{2} \phi N_{ab} + \xi \epsilon_{ab} + \zeta_{ab} \quad (15)$$

and

$$\dot{n}_a = \mathcal{A}u_a + \alpha_a, \quad (16)$$

where

$$a_a \equiv \hat{n}_a, \quad \phi \equiv \delta_a n^a, \quad \xi \equiv \frac{1}{2}\epsilon^{ab}\delta_a n_b, \quad \zeta_{ab} \equiv \delta_{\{a}n_{b\}}, \quad \mathcal{A} \equiv n^a \dot{u}_a, \quad \alpha_a \equiv \dot{n}_{\bar{a}} \quad (17)$$

and

$$\epsilon_{ab} \equiv \eta_{abc}n^c \equiv u^d \eta_{dabc}n^c. \quad (18)$$

Here η_{dabc} is the totally anti-symmetric 4-dimensional volume element with $\eta_{0123} = \sqrt{|\det g_{ab}|}$.

2.2. Fundamental equations in 1+3 split for the irrotational case

The propagation and constraint equations are given in [57]. We will here only consider the case of a perfect fluid with vanishing vorticity. Imposing $\omega_{ab} = 0$ introduces only one new constraint

$$\eta^{abc}\tilde{\nabla}_b \dot{u}_c = 0, \quad (19)$$

but with a barytropic equation of state, $p = p(\mu)$ where p is pressure and μ energy density in the rest frame of an observer, this equation is identically satisfied. From the Ricci identities one finds the following propagation equation for the expansion

$$\dot{\theta} - \tilde{\nabla}_a \dot{u}^a = -\frac{1}{3}\theta^2 + \dot{u}_a \dot{u}^a - 2\sigma^2 - \frac{1}{2}(\mu + 3p) + \Lambda, \quad (20)$$

where

$$\sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab} \quad (21)$$

and Λ is the cosmological constant.

The equation for the shear is

$$\dot{\sigma}^{<ab>} - \tilde{\nabla}^{<a}\dot{u}^{b>} = -\frac{2}{3}\theta\sigma^{ab} + \dot{u}^{<a}\dot{u}^{b>} - \sigma^{<a}_c\sigma^{b>c} - E^{ab}, \quad (22)$$

where $E_{ab} \equiv C_{acbd}u^c u^d$ is the electric part of the Weyl tensor. One also obtains the following constraints

$$\tilde{\nabla}_b \sigma^{ab} - \frac{2}{3}\tilde{\nabla}^a \theta = 0 \quad (23)$$

$$H^{ab} = (\text{curl } \sigma)^{ab} \equiv \eta^{cd< a}\tilde{\nabla}_c \sigma^{b>}_d, \quad (24)$$

where $H_{ab} \equiv \frac{1}{2}\eta_{ade}C^{de}_{bc}u^c$ is the magnetic part of the Weyl tensor.

From the twice contracted Bianchi identities one obtains

$$\dot{\mu} = -\theta(\mu + p) \quad (25)$$

$$\tilde{\nabla}_a p + (\mu + p)\dot{u}_a = 0 \quad (26)$$

and the remaining Bianchi identities give the following propagation equations

$$\dot{E}^{<ab>} = (\text{curl } H)^{ab} - \frac{1}{2}(\mu + p)\sigma^{ab} - \theta E^{ab} + 3\sigma^{<a}_c E^{b>c} + 2\eta^{cd< a}\dot{u}_c H^{b>}_d \quad (27)$$

$$\dot{H}^{<ab>} + (\text{curl } E)^{ab} = -\theta H^{ab} + 3\sigma^{<a}_c H^{b>c} - 2\eta^{cd< a}\dot{u}_c E^{b>}_d \quad (28)$$

and constraints

$$\tilde{\nabla}_b E^{ab} - \frac{1}{3} \tilde{\nabla}^a \mu - \eta^{abc} \sigma_{bd} H^d_c = 0 \quad (29)$$

$$\tilde{\nabla}_b H^{ab} + \eta^{abc} \sigma_{bd} E^d_c = 0 \quad (30)$$

respectively.

By differentiating the constraints with respect to "time" and using the commutators between the "time" and "spatial" derivatives it was shown in [60] that the constraints are propagated for irrotational dust and in [61] this result was extended to the barytropic case $p = p(\mu)$.

3. Kantowski-Sachs

Kantowski-Sachs cosmologies, [43], have a 4-dimensional isometry group acting multiply transitive on 3-spaces with topology $R \times S_2$, i.e. they are locally rotationally symmetric (LRS). With zero vorticity the line-element can be written as

$$ds^2 = -dt^2 + a_1^2(t) dz^2 + a_2^2(t) (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (31)$$

with the 4-velocity of matter given by $u = \frac{\partial}{\partial t}$ and the direction of anisotropy by $n = \frac{1}{a_1} \frac{\partial}{\partial z}$. The expansion is given by

$$\theta = \frac{\dot{a}_1}{a_1} + 2 \frac{\dot{a}_2}{a_2} \quad (32)$$

and in the tetrad

$$\omega^0 = dt, \quad \omega^1 = a_1 dz, \quad \omega^2 = a_2 d\vartheta, \quad \omega^3 = a_2 \sin \vartheta d\varphi, \quad (33)$$

the shear is given by

$$\Sigma \equiv \sigma_{11} = -2\sigma_{22} = -2\sigma_{33} = \frac{2}{3} \left(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} \right). \quad (34)$$

3.1. The evolution equations

Due to the LRS symmetry the shear and electric part of the Weyl tensor can be written as

$$\sigma_{ab} = \Sigma (n_a n_b - \frac{1}{2} N_{ab}). \quad (35)$$

and

$$E_{ab} = \mathcal{E} (n_a n_b - \frac{1}{2} N_{ab}) \quad (36)$$

respectively in terms of the anisotropy vector n^a and the projection operator N_{ab} . Given an equation of state and the cosmological constant, the Kantowski-Sachs models are completely determined in terms of shear, Σ , expansion, θ , and energy density, μ . The electric part of the Weyl tensor is given algebraically as

$$\mathcal{E} = -\frac{2}{3} \mu - \frac{2}{3} \Lambda - \Sigma^2 + \frac{2}{9} \theta^2 + \frac{1}{3} \Sigma \theta, \quad (37)$$

whereas

$$H_{ab} = \dot{u}_a = a_a = \phi = \xi = \zeta_{ab} = \mathcal{A} = \alpha_a = 0. \quad (38)$$

From the equations in section 2.2 the following evolution equations are obtained

$$\dot{\Sigma} = \frac{2}{3}\mu + \frac{2}{3}\Lambda + \frac{1}{2}\Sigma^2 - \Sigma\theta - \frac{2}{9}\theta^2 \quad (39)$$

$$\dot{\mu} = -\theta(\mu + p) \quad (40)$$

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \frac{1}{2}(\mu + 3p - 2\Lambda) - \frac{3}{2}\Sigma^2. \quad (41)$$

Alternatively one of the equations can be replaced by

$$\dot{K} = -\left(\frac{2}{3}\theta - \Sigma\right)K \quad (42)$$

where K , given by

$$K = \mu + \Lambda + \frac{3}{4}\Sigma^2 - \frac{1}{3}\theta^2 > 0, \quad (43)$$

is the curvature of the 2-spheres S_2 .

3.2. Dynamical system analysis of background

Dynamical system analysis of the Kantowski-Sachs models with positive cosmological constant were done in, e.g. [62, 24]. We here follow the notation in [24]. The relations between their variables and ours are given by

$$\begin{aligned} D &= \sqrt{\frac{1}{3}\mu + \frac{1}{3}\Lambda + \frac{1}{4}\Sigma^2}, \quad Q_0 = \frac{\theta}{3D}, \\ Q_+ &= -\frac{\Sigma}{2D}, \quad \tilde{\Omega}_\Lambda = \frac{\Lambda}{3D^2}, \quad \Omega_D = \frac{\mu}{2D^2}. \end{aligned} \quad (44)$$

The equilibrium points are given by

- $\pm F$: Flat Friedmann: $\Lambda = \Sigma = 0$, $\mu = \theta^2/3$, $Q_0 = \pm 1$, saddle points
- $+K_\pm$: Kasner: $\Lambda = \mu = 0$, $\Sigma = \mp \frac{2}{3}\theta$, $Q_0 = 1$, $Q_+ = \pm 1$, source points
- $-K_\pm$: Kasner: $\Lambda = \mu = 0$, $\Sigma = \pm \frac{2}{3}\theta$, $Q_0 = -1$, $Q_+ = \pm 1$, sink points
- $\pm dS$: de Sitter: $\mu = \Sigma = 0$, $\theta = \pm\sqrt{3\Lambda}$, sink/source points
- $\pm X$: $\mu = 0$, $\theta = \pm\sqrt{\Lambda}$, $\Sigma = \pm\frac{2}{3}\sqrt{\Lambda}$, saddle points.

The points $\pm X$ are exact vacuum solutions with metrics [24]

$$ds^2 = -dt^2 + e^{\pm 2\sqrt{\Lambda}t} dz^2 + \frac{1}{\Lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (45)$$

3.3. Vacuum solutions

In the vacuum case the equations can be integrated completely. If the curvature of S_2 , K , is a constant it follows that $\Sigma = \frac{2}{3}\theta$ (or $K = 0$). The system (39)-(42) then reduces to $\dot{\theta} = \Lambda - \theta^2$ with solutions

$$\theta = \sqrt{\Lambda} \frac{Ce^{\sqrt{\Lambda}t} - e^{-\sqrt{\Lambda}t}}{Ce^{\sqrt{\Lambda}t} + e^{-\sqrt{\Lambda}t}}, \quad (46)$$

where C is a constant of integration (and $\theta = \sqrt{\Lambda}$, corresponding to the critical point $+X$). By redefining the time coordinate $t \rightarrow t + t_0$, these solutions can be rewritten as one of the following two

$$\theta = \sqrt{\Lambda} \frac{\sinh(\sqrt{\Lambda}t)}{\cosh(\sqrt{\Lambda}t)}, \quad \theta = \sqrt{\Lambda} \frac{\cosh(\sqrt{\Lambda}t)}{\sinh(\sqrt{\Lambda}t)}. \quad (47)$$

($C = 0$ gives the critical point ${}_+X$ with $\theta = -\sqrt{\Lambda}$.) The corresponding line-element is given by

$$ds^2 = -dt^2 + f^2(t)dz^2 + \frac{1}{\Lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (48)$$

where $f(t)$ for the first solution is given by

$$f(t) = a_0 \cosh(\sqrt{\Lambda}t). \quad (49)$$

These spacetimes experience a bounce in the z -direction at $t = 0$ (and are non-expanding in the perpendicular directions). They start at the critical point ${}_+X$ and end at ${}_+X$. For the other solution one gets

$$f(t) = a_0 \sinh(\sqrt{\Lambda}t) \quad (50)$$

and hence these solutions are singular for $t = 0$. They start at the critical point ${}_+K_-$ (Kasner) and ends at ${}_+X$, or (for large negative t) start at ${}_+X$ and end at ${}_+K_+$.

If K is not a constant it can be used as the independent coordinate. The system (39)-(42) for $\mu = p = 0$, together with the constraint (43), then reduces to (see [63] for details)

$$\theta = \pm \frac{2\Lambda - 4K + 6MK^{3/2}}{2\sqrt{2MK^{3/2} - K + \Lambda/3}}, \quad \Sigma = \pm \frac{2K - 6MK^{3/2}}{3\sqrt{2MK^{3/2} - K + \Lambda/3}}, \quad (51)$$

where M is a constant of integration. Changing independent coordinate through $K = 1/T^2$ the corresponding line-element becomes

$$ds^2 = -\frac{dT^2}{\left(\frac{2M}{T} - 1 + \frac{\Lambda}{3}T^2\right)} + \left(\frac{2M}{T} - 1 + \frac{\Lambda}{3}T^2\right) dz^2 + T^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (52)$$

If $A \equiv 2M/T - 1 + \Lambda T^2/3 > 0$ this is the space-homogeneous region of the Schwarzschild-de Sitter metric [64], and $M = 0$ gives de Sitter space. With $\Lambda M^2/3 > 1/27$ the requirement $A > 0$ is satisfied for all positive T if $M > 0$ and the metric starts at the critical point ${}_+K_+$ ($T = 0$) and ends at ${}_+dS$ (de Sitter). $A > 0$ also in the region $(-\infty, T_0)$, where $T_0 < 0$ is the only real solution of $A = 0$. These spacetimes start at ${}_+dS$ and ends at ${}_+K_+$. If $M < 0$ the situation is reversed so that for $T < 0$ there is a class of solutions starting at ${}_+dS$ for large negative T and ending at ${}_+K_-$ for $T = 0$, and another class starting at ${}_+K_-$ for a positive T_0 and ending at ${}_+dS$ for large positive T .

For $0 < \Lambda M^2/3 < 1/27$ and $M > 0$ there is one class of solutions starting at ${}_+dS$ at negative infinity and ending at ${}_+K_+$ for a negative T_0 , another class starting at ${}_+K_+$ for $T = 0$ and ending at ${}_+K_+$ for a positive T_1 and a third class starting at ${}_+K_-$ for a positive T_2 and ending at ${}_+dS$ for large positive T . Similarly, for negative M , there is one class going from ${}_+dS$ to ${}_+K_+$, a second from ${}_+K_-$ to ${}_+K_-$ and a third from ${}_+K_-$ to ${}_+dS$.

3.4. Solutions with matter

The general dust solutions in terms of elliptic functions were found in [65, 66]. A few of them can be given in terms of elementary functions, see e.g. [66, 67]. For certain choices of the parameters in [66] physically reasonable solutions starting from a pancake like singularity can be obtained. These solutions end in expanding de Sitter. The solution [67] starts in ${}_+F$ and ends in ${}_+X$.

We here consider approximate solutions with $\mu \sim p \ll \Lambda$. To first order the dependent quantities are written as $\theta = \theta_0 + \theta_1$, $\Sigma = \Sigma_0 + \Sigma_1$ and $\mu = \mu_1$. Assuming a linear equation of state $p = (\gamma - 1)\mu$ and $\Sigma_0 = \frac{2}{3}\theta_0$ for the background vacuum metric, the following first order system

$$\dot{\mu}_1 = -\theta_0 \gamma \mu_1 \quad (53)$$

$$\dot{\Sigma}_1 - \frac{2}{3}\dot{\theta}_1 = \gamma \mu_1 + \theta_0 \left(\Sigma_1 - \frac{2}{3}\theta_1 \right) \quad (54)$$

$$\dot{\Sigma}_1 + \frac{5}{6}\dot{\theta}_1 = \frac{1}{4}(6 - 5\gamma)\mu_1 - 2\theta_0 \left(\Sigma_1 + \frac{5}{6}\theta_1 \right) \quad (55)$$

is obtained. In appendix Appendix A solutions around the bounce metric (49) and the critical points $\pm X$, (45), are given. Since for all of these solutions some terms grow unbounded, they are only valid for limited time intervals. Nevertheless they can be used to check our numerical codes for shorter time intervals in the case with densities that are initially low, and also for finding suitable starting conditions.

3.5. Bouncing solutions

Bouncing FLRW universes containing dust and radiation with cosmological constant are excluded by observations [68], see also Appendix E. This relies on a very simple argument. If quasars are observed at redshift $z > 4$, the dust component Ω_m cannot overcome the value 0.02, otherwise the bounce would occur at a smaller redshift than 4. We now consider whether a similar argument also holds for bouncing Kantowski-Sachs universes with cosmological constant. The bounce occurs either

i) in the direction of anisotropy at a_{1*} for which $\dot{a}_{1*} = 0$ and $\ddot{a}_{1*} > 0$

or

ii) in the perpendicular direction at a_{2*} for which $\dot{a}_{2*} = 0$ and $\ddot{a}_{2*} > 0$.

By defining the average scale factor a through

$$\theta = 3 \frac{\dot{a}}{a} . \quad (56)$$

and using (32) one obtains $a^3 = a_1 a^2$. If we assume that the universe is filled with dust ($\gamma = 1$) and radiation ($\gamma = 4/3$) Eq. (40) is satisfied by the following density

$$\mu = \mu_m + \mu_r = \mu_{m0} \left(\frac{a_0}{a} \right)^3 + \mu_{r0} \left(\frac{a_0}{a} \right)^4 , \quad (57)$$

where the subscript 0 denotes present values. We then introduce the usual dimensionless parameters

$$\Omega_m = 3\mu_{m0}/\theta_0^2 , \quad \Omega_r = 3\mu_{r0}/\theta_0^2 \quad \text{and} \quad \Omega_\Lambda = 3\Lambda/\theta_0^2 . \quad (58)$$

Case i): Since $\theta/3 + \Sigma = \dot{a}_1/a_1$, we have $0 < \dot{\theta}_*/3 + \dot{\Sigma}_*$ and by using Eqs. (39), (41) and the notations (58) this leads to

$$0 < \frac{3}{\theta_0^2} \frac{\ddot{a}_{1*}}{a_{1*}} = \frac{\Omega_m}{2} \left(\frac{a_0}{a_*} \right)^3 + \frac{\Omega_r}{3} \left(\frac{a_0}{a_*} \right)^4 + \Omega_\Lambda . \quad (59)$$

Eq. (43) gives

$$0 < \frac{3}{\theta_0^2 a_{2*}^2} + \frac{3}{\theta_0^2} \frac{\dot{a}_{2*}^2}{a_{2*}^2} = \Omega_m \left(\frac{a_0}{a_*} \right)^3 + \Omega_\Lambda \quad (60)$$

and another combination of Eqs. (39) and (41), $2\dot{\theta}_*/3 - \dot{\Sigma}_*$, gives

$$-\frac{6}{\theta_0^2} \frac{\ddot{a}_{2*}}{a_{2*}} = \Omega_m \left(\frac{a_0}{a_*} \right)^3 + \frac{4}{3} \Omega_r \left(\frac{a_0}{a_*} \right)^4. \quad (61)$$

The inequalities are always satisfied with a positive cosmological constant and the equations also give the values of \dot{a}_{2*} , \ddot{a}_{2*} and \ddot{a}_{1*} at the bounce. Therefore, we have no other constraints unless we can find some further integrals to the system (39)-(41).

We can also consider the equations (39), (41) and (43) at the present time t_0 . Neglecting radiation one obtains

$$-\frac{1}{3} q_{10} = \frac{1}{2} \Omega_m + \Omega_\Lambda - \frac{2H_{10}H_{20}}{3H_0^2} \quad (62)$$

$$-\frac{2}{3} q_{20} = -\Omega_m + \frac{2H_{10}H_{20}}{3H_0^2} \quad (63)$$

$$\frac{1}{3H_0^2 a_{20}^2} = \Omega_m + \Omega_\Lambda - 1 + \frac{(\Delta H_0)^2}{9H_0^2}, \quad (64)$$

where $H_0 = \dot{a}_0/a_0$ is the present value of the Hubble constant, $H_{10} = \dot{a}_{10}/a_{10}$, $H_{20} = \dot{a}_{20}/a_{20}$, $\Delta H_0 = H_{10} - H_{20}$, $q_{10} = -\ddot{a}_{10}/(a_{10}H_0^2)$ and $q_{20} = -\ddot{a}_{20}/(a_{20}H_0^2)$. How the quantities H_{10} , H_{20} , q_{10} and q_{20} relate to observations can be seen from the general expression for redshift (when both emitter (E) and receiver (R) have fixed spatial coordinates)

$$1+z = \frac{\nu_E}{\nu_R} = \frac{p_t(E)}{p_t(R)} \sqrt{\frac{g_{tt}(R)}{g_{tt}(E)}} \quad (65)$$

(see for example [69]). Here $p_a = g_{ab}dx^b/d\lambda$ (with affine parameter λ) denotes the covariant components of photon 4-momentum. By integrating the geodesic equations for the Kantowski-Sachs metric the following redshift formula is obtained for photons moving in the $\theta = \pi/2$ plane

$$1+z = \frac{a_0}{a} \left(\frac{\alpha_0}{\alpha} \right)^{2/3} \left[\frac{p_z^2 + p_\varphi^2 \alpha^2}{p_z^2 + p_\varphi^2 \alpha_0^2} \right]^{1/2}, \quad (66)$$

where $\alpha = a_1/a_2$ and p_z and p_φ are integration constants. For example, photons moving in the z -direction have a redshift $1+z = a_{10}/a_1$ whereas those moving along the 2-spheres have redshifts $1+z = a_{20}/a_2$.

In the isotropic limit $\Delta H_0 \rightarrow 0$, $\Delta q_0 = q_{10} - q_{20} \rightarrow 0$ and $1/a_{20}^2 \rightarrow 0$ the equations (62)-(64) agree with those of the flat FLRW model with cosmological constant ($q_0 \equiv \ddot{a}_0 a_0 / \dot{a}_0^2 = \Omega_m/2 - \Omega_\Lambda$, $1 = \Omega_m + \Omega_\Lambda$). Since there are indications of anisotropies, but with large uncertainties, both in the Hubble and deceleration parameters, see e.g. [70] and [71], it would be of interest to get better estimates of these parameters, even if present values do not seem to support Kantowski-Sachs models.

Case *ii*): From $K = 1/a_2^2$ we find $\ddot{K} = 2(-\ddot{a}_2/a_2 + 3\dot{a}_2^2/a_2^2)K$ and therefore $\ddot{K}_* < 0$. Then, taking the derivative of Eq. (42) to get another expression for \ddot{K} and using Eqs. (39) and (41) together with $\theta_* = 3\Sigma_*/2$, we find

$$\Omega_m < 0. \quad (67)$$

Hence a bounce in the scale factor a_2 is not possible, whereas a bounce in the direction of anisotropy cannot be excluded simply from this type of argument.

If we instead look at a bounce in the average scale factor at $a = a_{\#}$, so that $\theta_{\#} = 3\dot{a}_{\#}/a_{\#} = 0$ and $\ddot{a}_{\#} > 0$, the following inequality

$$\frac{2}{3H_0^2} \frac{1}{a_{2\#}^2} + \Omega_{\Lambda} < \frac{3}{2} \Omega_m \left(\frac{a_0}{a_{\#}} \right)^3 \quad (68)$$

can be derived, but no upper bound for the redshift at the bounce can be obtained from this either.

4. Perturbations on Kantowski-Sachs

In this section we calculate the equations governing the growth of density perturbations on a Kantowski-Sachs background to first order. The inhomogeneities will be described by quantities that are zero on the background, and hence are gauge invariant [18]. The primary variable is the density gradient

$$\mathcal{D}_a \equiv \frac{a\tilde{\nabla}_a \mu}{\mu}. \quad (69)$$

Here a is the average scale factor, defined in (56). The density fluctuations $\frac{\delta\mu}{\mu}$ on a length scale l are related to the quantity \mathcal{D}_a through $\frac{\delta\mu}{\mu} \sim (\mathcal{D}_a \mathcal{D}^a)^{1/2} l/a = (\mathcal{D}_a \mathcal{D}^a)^{1/2} l_0$, where $l_0 = l/a$ is the comoving dimensionless length scale. However, note that the quantity

$$\delta(x) \equiv \frac{\mu(x) - \bar{\mu}(x)}{\mu(x)} \quad (70)$$

depends on the identification between the fictitious background with density $\bar{\mu}(x)$ and the real universe with density $\mu(x)$ and can be given any value by changing the identification [57], whereas \mathcal{D}_a is gauge invariant.

To close the system three auxiliary quantities, that we choose as

$$\mathcal{Z}_a \equiv a\tilde{\nabla}_a \theta, \quad \mathcal{T}_a \equiv a\tilde{\nabla}_a \sigma^2 \quad \text{and} \quad \mathcal{S}_a \equiv a\tilde{\nabla}_a (\sigma^{ab} S_{ab}) \quad (71)$$

will be needed. Here the traceless part of the 3-Ricci tensor is given by

$$\begin{aligned} S_{ab} &\equiv {}^3R_{ab} - \frac{1}{3} {}^3R h_{ab} = -\dot{\sigma}_{\langle ab \rangle} - \theta \sigma_{ab} + \tilde{\nabla}_{\langle a} \dot{u}_{b \rangle} + \dot{u}_{\langle a} \dot{u}_{b \rangle} \\ &= E_{ab} + \sigma_{\langle a}^c \sigma_{b \rangle c} - \frac{1}{3} \theta \sigma_{ab}. \end{aligned} \quad (72)$$

In accordance with (9) σ_{ab} and S_{ab} can be decomposed as

$$\sigma_{ab} = \Sigma(n_a n_b - \frac{1}{2} N_{ab}) + 2\Sigma_{(a} n_{b)} + \Sigma_{ab} \quad (73)$$

and

$$S_{ab} = \tilde{S}(n_a n_b - \frac{1}{2} N_{ab}) + 2\tilde{S}_{(a} n_{b)} + \tilde{S}_{ab} \quad (74)$$

respectively. Note that S_{ab} to zeroth order is given in terms of the other quantities, but to first order is an independent quantity.

4.1. The first order equations

The propagation equations for the gradients are obtained by taking the gradients $\tilde{\nabla}_a$ of the propagation equations in section 2.2 and then using the commutator between "time" and "spatial" derivatives acting on a scalar, that to first order reduces to [57]

$$\tilde{\nabla}_a(\dot{f}) - (\tilde{\nabla}_a f)^\cdot = -\dot{u}_a \dot{f} + \frac{1}{3}\theta \tilde{\nabla}_a f + \sigma_a{}^c \tilde{\nabla}_c f. \quad (75)$$

In Appendix B some details of the calculations are given. Finally the following first order system

$$\dot{\mathcal{D}}_a = \frac{\theta p}{\mu} \mathcal{D}_a - \frac{3}{2} \Sigma n_a n^c \mathcal{D}_c + \frac{1}{2} \Sigma \mathcal{D}_a - \mathcal{Z}_a \left(1 + \frac{p}{\mu}\right) \quad (76)$$

$$\begin{aligned} \dot{\mathcal{Z}}_a = & -\frac{2}{3}\theta \mathcal{Z}_a - \frac{1}{2}\mu \mathcal{D}_a - \frac{3}{2}\Sigma n_a n^c \mathcal{Z}_c + \frac{1}{2}\Sigma \mathcal{Z}_a - 2\mathcal{T}_a + \frac{3}{2}\frac{\mu p'}{\mu + p} \left(\tilde{S} + \frac{3}{2}\Sigma^2\right) \mathcal{D}_a - \\ & \frac{\mu p'}{\mu + p} \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{D}_b \end{aligned} \quad (77)$$

$$\begin{aligned} \dot{\mathcal{T}}_a = & -2\theta \mathcal{T}_a - \frac{3}{2}\Sigma n_a n^c \mathcal{T}_c + \frac{1}{2}\Sigma \mathcal{T}_a - \mathcal{S}_a - \frac{3}{2}\Sigma^2 \mathcal{Z}_a + \frac{3}{2}\Sigma \left(\tilde{S} + \theta \Sigma\right) \frac{\mu p'}{\mu + p} \mathcal{D}_a - \\ & \frac{\mu p'}{\mu + p} \Sigma \left(\frac{3}{2} n_c n^b \tilde{\nabla}_a \tilde{\nabla}^c \mathcal{D}_b - \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{D}_b\right) \end{aligned} \quad (78)$$

$$\begin{aligned} \dot{\mathcal{S}}_a = & \left(\Sigma^2 + 2\frac{\tilde{S}^2}{\Sigma^2}\right) \mathcal{T}_a + \left(\frac{5}{2}\Sigma - \frac{5}{3}\theta - 2\frac{\tilde{S}}{\Sigma}\right) \mathcal{S}_a - \frac{3}{2}\Sigma n_a n^c \mathcal{S}_c - \Sigma \left(\frac{5}{2}\tilde{S} + \frac{2}{3}\Sigma\theta\right) \mathcal{Z}_a + \\ & \frac{p'\mu}{\mu + p} \tilde{S} \left(\frac{5}{2}\theta \Sigma + \frac{3}{2}\tilde{S} - \frac{3}{2}\Sigma^2\right) \mathcal{D}_a + \mu \Sigma^2 \mathcal{D}_a + \frac{p'\mu}{\mu + p} \left[\frac{1}{2} \left(\tilde{S} - \frac{1}{3}\theta \Sigma + 2\Sigma^2\right) \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{D}_b - \right. \\ & \left. \frac{3}{2} \left(\tilde{S} - \frac{1}{3}\theta \Sigma + \Sigma^2\right) n^b n^c \tilde{\nabla}_a \tilde{\nabla}_b \mathcal{D}_c\right] + \frac{3}{2}\Sigma n^b n^c \tilde{\nabla}_a \tilde{\nabla}_b \mathcal{Z}_c - \frac{1}{2}\Sigma \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{Z}_b - \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{T}_b, \end{aligned} \quad (79)$$

where

$$\tilde{S} = -\frac{2}{3}\mu - \frac{2}{3}\Lambda - \frac{1}{2}\Sigma^2 + \frac{2}{9}\theta^2 = -\frac{2}{3}K < 0 \quad (80)$$

to zeroth order and $p' \equiv dp/d\mu$, is obtained. As seen two apparently singular terms, $2\tilde{S}^2/\Sigma^2 \mathcal{T}_a$ and $-2\tilde{S}/\Sigma \mathcal{S}_a$, appear in (79). Near points where $\Sigma = 0$ it is hence suitable to remove these terms by changing the dependent variable \mathcal{T}_a to $\tilde{\mathcal{T}}_a$ through

$$\mathcal{T}_a = \Sigma^2 \tilde{\mathcal{T}}_a + \frac{\Sigma}{\tilde{S}} \mathcal{S}_a. \quad (81)$$

Instead factors $1/\tilde{S}$ and $1/\tilde{S}^2$ will be introduced, but since $\tilde{S} = -2K/3 < 0$ these will be well behaved.

4.2. The projected equations

The equations (76)-(79) can be decomposed into two sets by projecting with n_a and N_{ab} respectively. For Kantowski-Sachs, for which (38) holds, it follows from equations (16) and (15) for the derivatives \dot{n}_a and $\tilde{\nabla}_a n_b$ of n_a that

$$\dot{n}_a = \tilde{\nabla}_a n_b = 0. \quad (82)$$

Since $\dot{u}^a = 0$ and $h_{ab} u^a = 0$ it also follows that the derivatives of N_{ab} , \dot{N}_{ab} and $\tilde{\nabla}_c N_{ab}$, become zero:

$$\dot{N}_{ab} = u^c \nabla_c (g_{ab} + u_a u_b + n_a n_b) = \dot{u}_a u_b + \dot{u}_b u_a = 0$$

$$\tilde{\nabla}_c N_{ab} = \tilde{\nabla}_c(u_a u_b) = h_c^f h_a^d h_b^e (u_d \nabla_f u_e + u_e \nabla_f u_d) = 0.$$

Hence, since we only need n_a and N_{ab} to zeroth order, we can just let n^a and N_{ab} "pass through" the derivatives when projecting equations (76)-(79).

With the definitions

$$\mathcal{D} \equiv \mathcal{D}_a n^a, \mathcal{Z} \equiv \mathcal{Z}_a n^a, \mathcal{T} \equiv \mathcal{T}_a n^a, \mathcal{S} \equiv \mathcal{S}_a n^a \quad (83)$$

the equations projected along n^a become:

$$\dot{\mathcal{D}} = \left(\frac{\theta p}{\mu} - \Sigma \right) \mathcal{D} - \left(1 + \frac{p}{\mu} \right) \mathcal{Z} \quad (84)$$

$$\dot{\mathcal{Z}} = -\frac{1}{2} \mu \mathcal{D} - \left(\frac{2}{3} \theta + \Sigma \right) \mathcal{Z} - 2\mathcal{T} + \frac{3}{2} \frac{\mu p'}{\mu + p} \left(\tilde{S} + \frac{3}{2} \Sigma^2 \right) \mathcal{D} - \frac{\mu p'}{\mu + p} n^a \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{D}_b \quad (85)$$

$$\begin{aligned} \dot{\mathcal{T}} = & -(2\theta + \Sigma) \mathcal{T} - \frac{3}{2} \Sigma^2 \mathcal{Z} + \frac{3}{2} \frac{\mu p'}{\mu + p} \Sigma \left(\tilde{S} + \theta \Sigma \right) \mathcal{D} - \mathcal{S} - \frac{\mu p'}{\mu + p} \Sigma \left(\frac{3}{2} n^a \tilde{\nabla}_a n^b \tilde{\nabla}_b \mathcal{D} - \right. \\ & \left. \frac{1}{2} n^a \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{D}_b \right) \end{aligned} \quad (86)$$

$$\begin{aligned} \dot{\mathcal{S}} = & \left(\Sigma^2 + 2 \frac{\tilde{S}^2}{\Sigma^2} \right) \mathcal{T} + \left(\Sigma - \frac{5}{3} \theta - 2 \frac{\tilde{S}}{\Sigma} \right) \mathcal{S} - \Sigma \left(\frac{5}{2} \tilde{S} + \frac{2}{3} \Sigma \theta \right) \mathcal{Z} + \frac{p' \mu}{\mu + p} \tilde{S} \left(\frac{5}{2} \theta \Sigma + \frac{3}{2} \tilde{S} - \frac{3}{2} \Sigma^2 \right) \mathcal{D} + \\ & \mu \Sigma^2 \mathcal{D} + \frac{p' \mu}{\mu + p} \left[\frac{1}{2} \left(\tilde{S} - \frac{1}{3} \theta \Sigma + 2 \Sigma^2 \right) n^a \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{D}_b - \frac{3}{2} \left(\tilde{S} - \frac{1}{3} \theta \Sigma + \Sigma^2 \right) n^a \tilde{\nabla}_a n^b \tilde{\nabla}_b \mathcal{D} \right] - \\ & n^a \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{T}_b + \frac{3}{2} \Sigma n^a \tilde{\nabla}_a n^b \tilde{\nabla}_b \mathcal{Z} - \frac{1}{2} \Sigma n^a \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{Z}_b. \end{aligned} \quad (87)$$

The terms $\tilde{\nabla}^b \mathcal{D}_b$ etc. can be decomposed as

$$\tilde{\nabla}^b \mathcal{D}_b = n^b \tilde{\nabla}_b \mathcal{D} + \delta^b \mathcal{D}_{\bar{b}} \quad (88)$$

to first order.

The orthogonal equations, obtained by projecting with N_{ab} , can be found in Appendix C.

4.3. Scalar equations

To treat the spatial derivatives appearing in the equations we will do a harmonic decomposition. For this purpose it is suitable to get the spatial derivatives in the form of two Laplace-like operators,

$$\delta^2 \equiv \delta_a \delta^a \quad \text{and} \quad \hat{\Delta} \equiv n^a \tilde{\nabla}_a n^b \tilde{\nabla}_b \quad (89)$$

acting on our variables. To obtain this we define new variables

$$\hat{\mathcal{D}} \equiv n^a \tilde{\nabla}_a \mathcal{D} \quad \text{and} \quad \mathcal{P} \equiv \delta^a \mathcal{D}_{\bar{a}} \quad (90)$$

and similarly for the other variables. We then act on the system (84)-(87) with the operator $n^a \tilde{\nabla}_a$ and use the commutation relation

$$\hat{\Psi} - \dot{\Psi} = \left(\frac{1}{3} \theta + \Sigma \right) \hat{\Psi}, \quad (91)$$

[59], that holds to first order. To remove the singular terms $2\tilde{S}^2/\Sigma^2 \hat{\mathcal{T}}$ and $-2\tilde{S}/\Sigma \hat{\mathcal{S}}$ we now also make the aforementioned change of the dependent variables $\hat{\mathcal{T}}$ and \mathcal{T} to $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{T}}$ through

$$\hat{\mathcal{T}} = \Sigma^2 \tilde{\mathcal{T}} + \frac{\Sigma}{\tilde{S}} \hat{\mathcal{S}} \quad \text{and} \quad \mathcal{T} = \Sigma^2 \tilde{\mathcal{T}} + \frac{\Sigma}{\tilde{S}} \mathcal{S} \quad (92)$$

respectively. The system for the hat variables then becomes

$$\dot{\hat{\mathcal{D}}} = \left[\theta \left(\frac{p}{\mu} - \frac{1}{3} \right) - 2\Sigma \right] \hat{\mathcal{D}} - \left(1 + \frac{p}{\mu} \right) \hat{\mathcal{Z}} \quad (93)$$

$$\dot{\hat{\mathcal{Z}}} = -(\theta + 2\Sigma) \hat{\mathcal{Z}} + \left[-\frac{1}{2}\mu + \frac{3}{2} \frac{\mu p'}{\mu + p} \left(\tilde{S} + \frac{3}{2}\Sigma^2 \right) \right] \hat{\mathcal{D}} - 2 \frac{\Sigma}{\tilde{S}} \hat{\mathcal{S}} - 2\Sigma^2 \tilde{\mathcal{T}} - \frac{\mu p'}{\mu + p} \hat{\Delta} [\hat{\mathcal{D}} + \mathcal{P}] \quad (94)$$

$$\begin{aligned} \dot{\tilde{\mathcal{T}}} = & - \left(\frac{1}{3}\theta + 2\Sigma + \frac{\Sigma^3}{\tilde{S}} \right) \tilde{\mathcal{T}} - \left(\frac{\Sigma^2}{\tilde{S}^2} + \frac{1}{\tilde{S}} \right) \hat{\mathcal{S}} - \left[\frac{\Sigma\mu}{\tilde{S}} + \frac{\mu p'}{\mu + p} \left(\theta - \frac{3}{2}\Sigma \right) \right] \hat{\mathcal{D}} + \left(1 + \frac{2}{3} \frac{\Sigma\theta}{\tilde{S}} \right) \hat{\mathcal{Z}} + \\ & \frac{\mu p'}{\mu + p} \frac{1}{\tilde{S}} \left[\left(\frac{1}{2}\Sigma - \frac{1}{3}\theta \right) \hat{\Delta} \hat{\mathcal{D}} - \left(\Sigma - \frac{1}{6}\theta \right) \hat{\Delta} (\mathcal{P}) \right] - \frac{1}{\tilde{S}} \hat{\Delta} (\hat{\mathcal{Z}} - \frac{1}{2}\mathcal{Z}) + \frac{\Sigma}{\tilde{S}} \hat{\Delta} (\tilde{\mathcal{T}} + \mathcal{T}) + \\ & \frac{1}{\tilde{S}^2} \hat{\Delta} (\hat{\mathcal{S}} + \mathcal{S}) \end{aligned} \quad (95)$$

$$\begin{aligned} \dot{\hat{\mathcal{S}}} = & \left[\mu\Sigma^2 + \frac{\mu p'}{\mu + p} \tilde{S} \left(\frac{5}{2}\theta\Sigma + \frac{3}{2}\tilde{S} - \frac{3}{2}\Sigma^2 \right) \right] \hat{\mathcal{D}} - \left(\frac{2}{3}\theta\Sigma + \frac{5}{2}\tilde{S} \right) \Sigma \hat{\mathcal{Z}} + \left(\Sigma^4 + 2\tilde{S}^2 \right) \tilde{\mathcal{T}} + \\ & \left(\frac{\Sigma^3}{\tilde{S}} - 2\theta \right) \hat{\mathcal{S}} + \Sigma \hat{\Delta} \left(\hat{\mathcal{Z}} - \frac{1}{2}\mathcal{Z} \right) - \Sigma^2 \hat{\Delta} (\tilde{\mathcal{T}} + \mathcal{T}) + \frac{\mu p'}{\mu + p} \left[\left(-\tilde{S} + \frac{1}{3}\theta\Sigma - \frac{1}{2}\Sigma^2 \right) \hat{\Delta} \hat{\mathcal{D}} \right. \\ & \left. + \frac{1}{2} \left(\tilde{S} - \frac{1}{3}\theta\Sigma + 2\Sigma^2 \right) \hat{\Delta} (\mathcal{P}) \right] - \frac{\Sigma}{\tilde{S}} \hat{\Delta} (\hat{\mathcal{S}} + \mathcal{S}). \end{aligned} \quad (96)$$

By taking the 2-divergence of the system (C.1)-(C.4) and using the commutation relation

$$\delta^a \dot{\Psi}_{\bar{a}} - (\delta^a \Psi_{\bar{a}})^{\cdot} = \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma \right) \delta^a \Psi_{\bar{a}}, \quad (97)$$

[59], (where $\delta^a \Psi_{\bar{a}} = \Psi$ according to the above definition) a similar system, that can be found in Appendix C, is obtained for the slashed variables. As we will see in the next section, the hat and slash variables are closely related.

Since the scale factors $a_1(t)$ and $a_2(t)$ appear in the spatial derivatives, the time dependence of the variables $\hat{\mathcal{D}}$ and \mathcal{P} will go as \mathcal{D}/a_1 and $\mathcal{D}_{\bar{a}}/a_2$ respectively. This is most easily seen by calculating $\hat{\mathcal{D}}$ and \mathcal{P} in the tetrad (33), giving

$$\hat{\mathcal{D}} = \frac{1}{a_1} \frac{\partial \mathcal{D}}{\partial z} \quad \text{and} \quad \mathcal{P} = \frac{1}{a_2} \left(\frac{\partial D_{\bar{2}}}{\partial \vartheta} + \frac{1}{\sin \vartheta} \frac{\partial D_{\bar{3}}}{\partial \varphi} \right). \quad (98)$$

Hence the time dependences of the variables

$$\mathcal{D}_{\parallel} = a_1 \hat{\mathcal{D}} \quad \text{and} \quad \mathcal{D}_{\perp} = a_2 \mathcal{P} \quad (99)$$

give a better representation of the development of the relative density contrast $\frac{\delta\mu}{\mu}$.

4.4. Harmonic decomposition

We will use a harmonic decomposition

$$\Psi = \sum_{k_{\parallel}, k_{\perp}} \Psi_{k_{\parallel} k_{\perp}} P_{k_{\parallel}} Q_{k_{\perp}} \quad (100)$$

of the dependent variables. Here $P_{k_{\parallel}}$ satisfies

$$\hat{\Delta} P_{k_{\parallel}} = -\frac{k_{\parallel}^2}{a_1^2} P_{k_{\parallel}}, \quad \delta_a P_{k_{\parallel}} = \dot{P}_{k_{\parallel}} = 0 \quad (101)$$

where k_{\parallel} are the constant co-moving wave numbers in the direction of anisotropy and the scale factor in this direction, a_1 , is given by the zeroth order equation

$$\frac{\dot{a}_1}{a_1} = \frac{1}{3}\theta + \Sigma. \quad (102)$$

If a coordinate z is adopted to the 1-direction a possible choice for $P_{k_{\parallel}}$ is $P_{k_{\parallel}} = e^{ik_{\parallel}z}$ as can be seen by direct substitution into (101), using a tetrad, e.g. (33), adopted to the symmetries. Similarly the harmonics $Q_{k_{\perp}}$ are introduced on the 2-sheets as was done in [59]. They satisfy

$$\delta^2 Q_{k_{\perp}} = -\frac{k_{\perp}^2}{a_2^2} Q_{k_{\perp}}, \quad \hat{Q}_{k_{\perp}} = \dot{Q}_{k_{\perp}} = 0 \quad (103)$$

where the scale factor on the 2-sheets, a_2 , is obtained from

$$\frac{\dot{a}_2}{a_2} = \frac{1}{3}\theta - \frac{1}{2}\Sigma, \quad (104)$$

and k_{\perp} are the co-moving wave numbers in the perpendicular directions.

In the irrotational case there is a simple relation between the coefficients of the modes for the hat and slash scalars. We note that to first order $\hat{\mathcal{D}}$ and \mathcal{P} can be written as

$$\hat{\mathcal{D}} = \frac{a}{\mu} \hat{\Delta} \mu \quad \text{and} \quad \mathcal{P} = \frac{a}{\mu} \delta^2 \mu \quad (105)$$

respectively, with similar expressions for the other scalars. From the commutation relations in [59] it follows that the operators $\hat{\Delta}$ and δ^2 commute to first order when the vorticity is zero. Hence

$$\delta^2 \hat{\mathcal{D}} = \delta^2 \left(\frac{a}{\mu} \hat{\Delta} \mu \right) = \hat{\Delta} \left(\frac{a}{\mu} \delta^2 \mu \right) = \hat{\Delta} \mathcal{P} \quad (106)$$

holds to first order. Using the harmonic decomposition (100) and equations (101) and (103), one has

$$\delta^2 \hat{\mathcal{D}} = - \sum_{k_{\parallel}, k_{\perp}} \hat{\mathcal{D}}_{k_{\parallel} k_{\perp}} P_{k_{\parallel}} \frac{k_{\perp}^2}{a_2^2} Q_{k_{\perp}} = - \sum_{k_{\parallel}, k_{\perp}} \mathcal{P}_{k_{\parallel} k_{\perp}} \frac{k_{\parallel}^2}{a_1^2} P_{k_{\parallel}} Q_{k_{\perp}} = \hat{\Delta} \mathcal{P} \quad (107)$$

with similar expressions for the other scalars. Hence we find the following relations

$$\Psi_{k_{\parallel} k_{\perp}} = \left(\frac{k_{\perp}}{k_{\parallel}} \right)^2 \left(\frac{a_1}{a_2} \right)^2 \hat{\Psi}_{k_{\parallel} k_{\perp}}, \quad (108)$$

where $\hat{\Psi}$ is $\hat{\mathcal{D}}$, $\hat{\mathcal{Z}}$, $\hat{\mathcal{T}}$ or $\hat{\mathcal{S}}$ and Ψ is \mathcal{P} , \mathcal{Z} , \mathcal{T} or \mathcal{S} , between the coefficients of the different modes. In the following we will often suppress the subscripts $k_{\parallel} k_{\perp}$ when it is obvious that we are referring to harmonic coefficients.

5. Perturbations of analytical solutions

In this section we summarize some results on perturbations around a few of the exact vacuum solutions from section 3.3 in the infinite wavelength ($k = 0$) limit. More details are given in appendix Appendix D. It might seem unphysical to consider density perturbations in a vacuum solution, but the solutions might approximate perturbations around some background metrics with $p \ll \mu \ll \Lambda$. Furthermore, the analytical results are useful for a comparison with results of the numerical codes. This also gives a method of identifying suitable initial conditions.

5.0.1. Perturbations of $+X$ With the $+X$ -solution as background one finds the following fourth order equation for the density gradients $\hat{\mathcal{D}}$ in the direction of anisotropy

$$\hat{\mathcal{D}}^{(4)} + \frac{23}{3}\sqrt{\Lambda}\hat{\mathcal{D}}^{(3)} + \frac{59}{3}\Lambda\ddot{\hat{\mathcal{D}}} + \frac{545}{27}\Lambda^{3/2}\dot{\hat{\mathcal{D}}} + \frac{550}{81}\Lambda^2\hat{\mathcal{D}} = 0 \quad (109)$$

if the harmonic numbers are put to zero (corresponding to large wavelengths of the perturbations). The solution \mathcal{D}_{\parallel} , i.e. $\hat{\mathcal{D}}$ multiplied with the scale factor $a_1 \propto e^{\sqrt{\Lambda}t}$, is given by

$$\mathcal{D}_{\parallel} = A_1 e^{-8\sqrt{\Lambda}t/3} + A_2 e^{\sqrt{\Lambda}t/3} + (A_3 t + A_4) e^{-2\sqrt{\Lambda}t/3}, \quad (110)$$

where A_1, A_2, A_3 and A_4 are integration constants (or slowly varying functions over space if the wave number is not exactly zero). Hence there is one growing and three decaying modes.

The solution for the density gradients $\mathcal{D}_{\perp} = a_2 \mathcal{P}$ in the perpendicular directions is then obtained from $\mathcal{D}_{\perp} = a_2 \left(\frac{a_1}{a_2}\right)^2 \hat{\mathcal{D}} \propto e^{2\sqrt{\Lambda}t} \hat{\mathcal{D}}$ as

$$\mathcal{D}_{\perp} = B_1 e^{-5\sqrt{\Lambda}t/3} + B_2 e^{4\sqrt{\Lambda}t/3} + (B_3 t + B_4) e^{\sqrt{\Lambda}t/3}. \quad (111)$$

As seen, in these directions that are not expanding, the modes are growing faster.

5.0.2. Perturbations of vacuum bounce solution With the vacuum bounce solution (49) as background it is suitable to change the independent variable to θ through

$$\dot{\psi} = \frac{d\psi}{d\theta} \dot{\theta} = \frac{d\psi}{d\theta} (\Lambda - \theta^2).$$

The following equation for the density gradient $\hat{\mathcal{D}}$ in the direction of anisotropy

$$\begin{aligned} \frac{d^4 \hat{\mathcal{D}}}{d\theta^4} + \frac{(7\theta^2 + 6\Lambda)}{3\theta(\theta^2 - \Lambda)} \frac{d^3 \hat{\mathcal{D}}}{d\theta^3} - \frac{(7\theta^4 - 6\Lambda\theta^2 - 6\Lambda^2)}{3\theta^2(\theta^2 - \Lambda)^2} \frac{d^2 \hat{\mathcal{D}}}{d\theta^2} + \frac{5(8\theta^4 - 45\Lambda\theta^2 + 18\Lambda^2)}{27\theta(\theta^2 - \Lambda)^3} \frac{d\hat{\mathcal{D}}}{d\theta} \\ - \frac{(-735\Lambda\theta^4 + 495\Lambda^2\theta^2 - 40\theta^6 - 270\Lambda^3)}{81\theta^2(\theta^2 - \Lambda)^4} \hat{\mathcal{D}} = 0 \end{aligned} \quad (112)$$

is then obtained in the long wavelength limit. The solution $\mathcal{D}_{\parallel} = a_1 \hat{\mathcal{D}}$, where $a_1 \propto 1/\sqrt{\Lambda - \theta^2}$, is then given by

$$\begin{aligned} \mathcal{D}_{\parallel} = (A_1 + A_2\theta) (\Lambda - \theta^2)^{1/3} + A_3\theta (\Lambda - \theta^2)^{-1/6} + A_4 (\Lambda - \theta^2)^{1/3} \times \\ \left[\frac{1}{2} \ln \left(1 - \frac{\theta^2}{\Lambda} \right) - \frac{\theta}{4\sqrt{\Lambda}} \ln \left(\frac{\sqrt{\Lambda} + \theta}{\sqrt{\Lambda} - \theta} \right) \frac{\theta}{\sqrt{\Lambda - \theta^2}} \arcsin \left(\frac{\theta}{\sqrt{\Lambda}} \right) \right] \end{aligned} \quad (113)$$

(assuming $\theta^2 < \Lambda$). Here $\theta = \sqrt{\Lambda} \tanh(\sqrt{\Lambda}t)$ and hence $\Lambda - \theta^2 = \Lambda / \cosh^2(\sqrt{\Lambda}t)$. The mode A_1 starts growing, obtains its largest value at the bounce and then starts decaying. The A_2 mode also starts growing, but reaches its maximum before the bounce. It then decays to zero magnitude at the bounce and after this passes through a new maximum before it eventually decays. The A_3 and A_4 modes initially decay, pass through zero at $t = 0$ and then grow unboundedly.

As in the previous case the growth of the density perturbations in the non-expanding directions are obtained from $\mathcal{D}_{\perp} = a_2 \mathcal{P} = a_2 \left(\frac{a_1}{a_2}\right)^2 \hat{\mathcal{D}} \propto \hat{\mathcal{D}}/(\Lambda - \theta^2)$

as

$$\mathcal{D}_\perp = (B_1 + B_2\theta)(\Lambda - \theta^2)^{-1/6} + B_3\theta(\Lambda - \theta^2)^{-2/3} + B_4(\Lambda - \theta^2)^{-1/6} \times \left[\frac{1}{2} \ln \left(1 - \frac{\theta^2}{\Lambda} \right) - \frac{\theta}{4\sqrt{\Lambda}} \ln \left(\frac{\sqrt{\Lambda} + \theta}{\sqrt{\Lambda} - \theta} \right) + \frac{\theta}{\sqrt{\Lambda - \theta^2}} \arcsin \left(\frac{\theta}{\sqrt{\Lambda}} \right) \right]. \quad (114)$$

These modes decay before the bounce and grow after. The modes B_3 and B_4 are close to zero for a longer period of time around the bounce.

For this solution perturbations with co-moving wave number $k_\parallel > 2a_1(t_0)\sqrt{\Lambda}$ cross the horizon in the anisotropy direction within a finite time, and one would expect a difference in behaviour between perturbations with wave numbers that are smaller or larger than $2a_1(t_0)\sqrt{\Lambda}$, respectively. This is also favored by the numerical analysis in section 6, where typically the perturbations in similar models show an oscillatory behaviour for wave numbers larger than this value.

6. Numerical solutions

The time evolution of the density gradients are solved for numerically in some representative cases. To find suitable backgrounds we have used the phase space analysis in [24] (see also section 3.2). Source points are given by the expanding Kasner metrics $+K_\pm$ and contracting de Sitter $-dS$, and sink points by the contracting Kasner metrics $-K_\pm$ and expanding de Sitter $+dS$. The plots shown below either start from expanding Kasner $+K_\pm$, or originates from them. They mainly end at expanding de Sitter $+dS$, being a more likely state of the late universe, but we also give examples of metrics starting from $+K_-$ and ending in $-K_-$. Saddle points are given by the flat Friedmann models $\pm F$ and the solutions $\pm X$, which were useful in finding solutions undergoing a bounce. This is related to that the vacuum bounce solution (49) starts in $-X$ and ends at $+X$. For each type of path in phase space one case with radiation and one with dust are given.

The system contains four different modes, \mathcal{D} , \mathcal{Z} , \mathcal{T} and \mathcal{S} , for each background and wave number. In most of the presented cases just an initial density perturbation is assumed, but in the two first cases we also show the effects of an initial pure shear perturbation. The behaviour for different types of initial perturbations has been checked, but the differences one can see are of the same types as those given between figures 2 and 3, and figures 4 and 5 respectively. The differences are largest in the dust case, where the density perturbations depend more strongly on wave number with initial shear and expansion perturbations than with an initial density perturbation due to that factors like $\hat{\Delta}(\tilde{\mathcal{T}} + \tilde{\mathcal{T}})$ appear as source terms in the equations. For each radiation case we show the evolution for the wave numbers $k = k_\parallel/a_{10} = k_\perp/a_{20} = 0, 1, 5$ and 20, including both sub horizontal and super horizontal perturbations. For dust, where the dependence on wave number is small for initial pure density perturbations, we only show the evolution for one wave number except for the case with initial shear perturbations.

Throughout this section we choose the cosmological constant $\Lambda = 1$. Other initial values are given in the figure captions or text. The code was tested against the results of section 5.

6.1. Solutions with a bounce

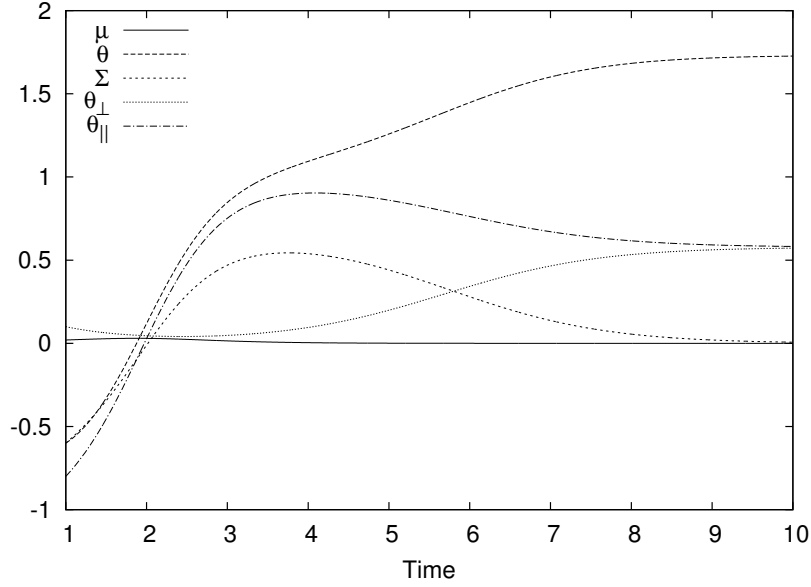


Figure 1. A radiation background that experiences a bounce. Initially it is close to $-X$ and at late times it approaches $+dS$. Initial values at $t_0 = 1$ are given by $\mu_0 = 0.02$, $\theta_0 = -0.6$, $\Sigma_0 = -0.6$, $\theta_{\parallel} = -0.8$ and $\theta_{\perp} = 0.1$.

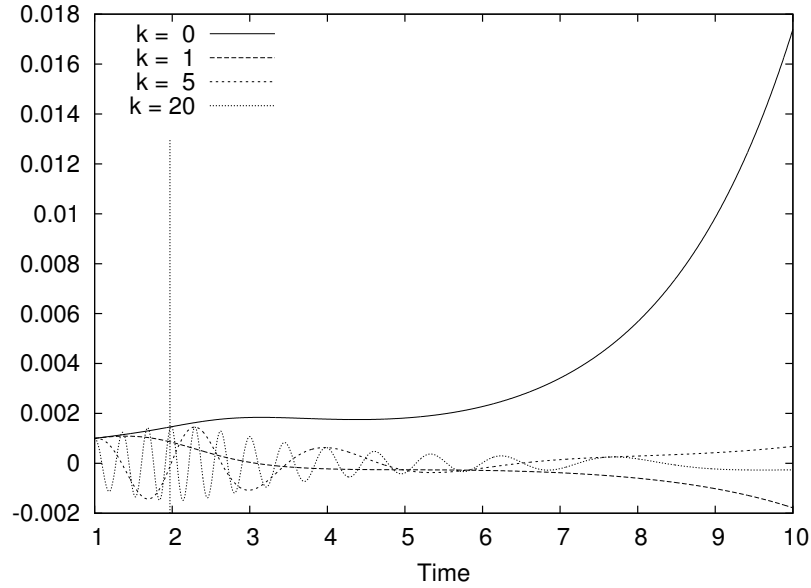


Figure 2. The growth of the density perturbations \mathcal{D}_{\parallel} in the background given by figure 1 for the wave numbers $k = k_{\parallel}/a_{10} = k_{\perp}/a_{20} = 0, 1, 5$ and 20 . Initially, at $t_0 = 1$, $\hat{\mathcal{D}} = \mathcal{P} = 0.001$. The time of the bounce is indicated with a dotted vertical line. (Here $a_{10} = a_1(t_0)$ and $a_{20} = a_2(t_0)$.)

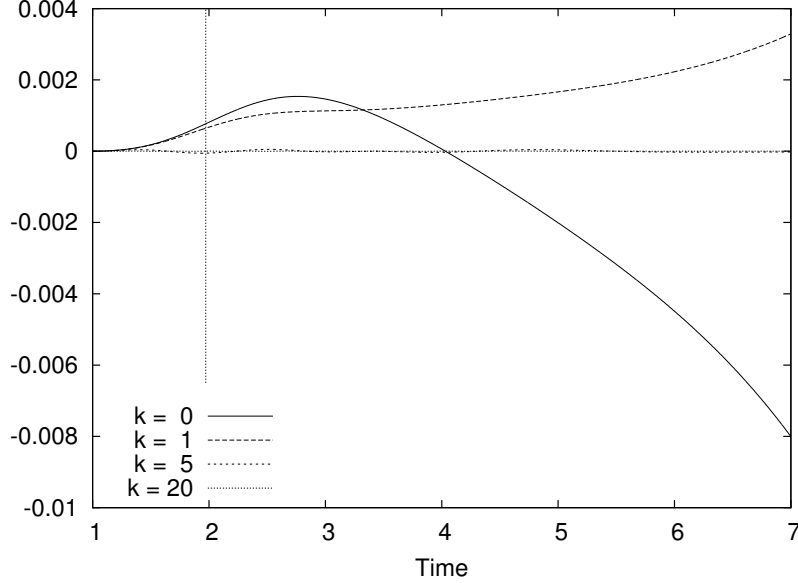


Figure 3. The growth of the density perturbations \mathcal{D}_{\parallel} in the background given by figure 1 for the wave numbers $k = k_{\parallel}/a_{10} = k_{\perp}/a_{20} = 0, 1, 5$ and 20. Initially, at $t_0 = 1$, $\tilde{\mathcal{T}} = \tilde{\mathcal{T}} = 0.001$. The time of the bounce is indicated with a dotted vertical line.

6.1.1. From $-X$ via $+X$ to $+dS$, radiation This background, where the equation of state is given by $p = \mu/3$ (radiation), passes through a state where the directions perpendicular to the direction of anisotropy have small and decreasing expansion rates, $\theta_{\perp} \equiv \dot{a}_2/a_2$, that become almost negligible for a period of time. After this the expansion starts again to finally reach a constant value $\theta_{\perp} = \sqrt{\Lambda/3}$. The anisotropy direction comes from a contracting state, goes through a bounce and then starts expanding. Eventually the expansion rate in this direction, $\theta_{\parallel} \equiv \dot{a}_1/a_1$, also approaches $\theta_{\parallel} = \sqrt{\Lambda/3}$, giving a total expansion rate of $\theta = \sqrt{3\Lambda}$. The initial state is close to the critical point $-X$ and for an intermediate period the solution is close to $+X$, but for large times the solution approaches the sink point de Sitter $+dS$. The evolutions of density, μ_0 , expansion, θ_0 , expansion in the anisotropy direction, θ_{\parallel} , expansion in one of the perpendicular directions, θ_{\perp} and shear Σ_0 are depicted in figure 1. Note that, as is found by running time backwards in the numerical code, that this solution can be seen as originating from the source point $+K_+$.

In figure 2 the growth of density perturbations for different values of the co-moving wave numbers k_{\parallel} and k_{\perp} is shown. Since from equation (108) $\mathcal{D}_{\perp} = \left(\frac{k_{\perp}}{k_{\parallel}}\right)^2 \frac{a_1}{a_2} \mathcal{D}_{\parallel}$ we only show \mathcal{D}_{\parallel} . The initial values of the density perturbations at $t_0 = 1$ are given by $\hat{\mathcal{D}} = \mathcal{P} = 0.001$ and the other quantities are initially put to zero, $\hat{\mathcal{Z}} = \mathcal{Z} = \tilde{\mathcal{T}} = \tilde{\mathcal{T}} = \hat{\mathcal{S}} = \mathcal{S} = 0$. For $k = 0$ the density gradient in the direction of anisotropy reaches a small maximum after the bounce and after this a small minimum before it starts growing unboundedly. In the perpendicular directions the gradient for the $k = 0$ case is roughly constant before it also starts growing. For higher values of the wave number k the density gradient in the anisotropy direction shows an oscillatory behaviour with

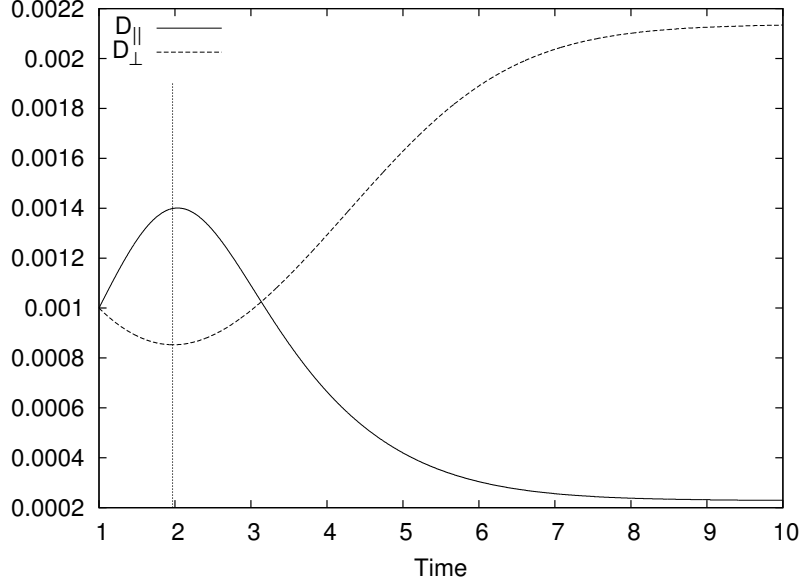


Figure 4. The growth of the density perturbations \mathcal{D}_{\parallel} and \mathcal{D}_{\perp} in the dust background of section 6.1.2 for the wave numbers $k = k_{\parallel}/a_{10} = k_{\perp}/a_{20} = 1$. Initially, at $t_0 = 1$, $\hat{\mathcal{D}} = \mathcal{P} = 0.001$ and $\mu_0 = 0.02$, $\theta_0 = -0.6$, $\Sigma_0 = -0.6$, $\theta_{\parallel} = -0.8$ and $\theta_{\perp} = 0.1$. The time of the bounce is indicated with a dotted vertical line.

an initially increasing amplitude that later on decreases, but does not fall off to zero. This behaviour with a local maximum in the density gradient at or slightly after the bounce in the bouncing direction seems to be typical. For corresponding k values in the perpendicular directions the oscillations initially have an approximately constant amplitude that with time slowly starts growing.

In Appendix E the corresponding growth of density perturbations in a closed and isotropic radiation filled universe undergoing a bounce (now in all three directions) is shown in figure E2. The local extrema seen in figure 2 are absent in the isotropic case, apart from a small dip for the $k = 0$ mode, whereas the behaviours for large times, when the universes in both cases approach de Sitter spacetime, are similar.

In figure 3 the growth of density perturbations for a case where the density perturbations initially are zero, $\hat{\mathcal{D}} = \mathcal{P} = 0$, is shown. Instead there is an initial shear perturbation $\hat{\mathcal{T}} = \mathcal{T} = 0.001$ (and $\hat{\mathcal{Z}} = \mathcal{Z} = \hat{\mathcal{S}} = \mathcal{S} = 0$). As seen the super horizon modes $k = 0, 1$ grows unboundedly, whereas for higher k -values the amplitudes remain small.

6.1.2. From $-X$ via $+X$ to $+dS$, dust The conditions here are similar to those in the previous section, but the equation of state is now given by dust, $p = 0$. With the same initial conditions for the background quantities as in the radiation case, their evolutions are close to those of the radiation case, see figure 1. First we show the time development of the density perturbations for the case when initially $\hat{\mathcal{D}} = \mathcal{P} = 0.001$ and $\hat{\mathcal{Z}} = \mathcal{Z} = \hat{\mathcal{T}} = \mathcal{T} = \hat{\mathcal{S}} = \mathcal{S} = 0$. Since in this case the evolution of the density

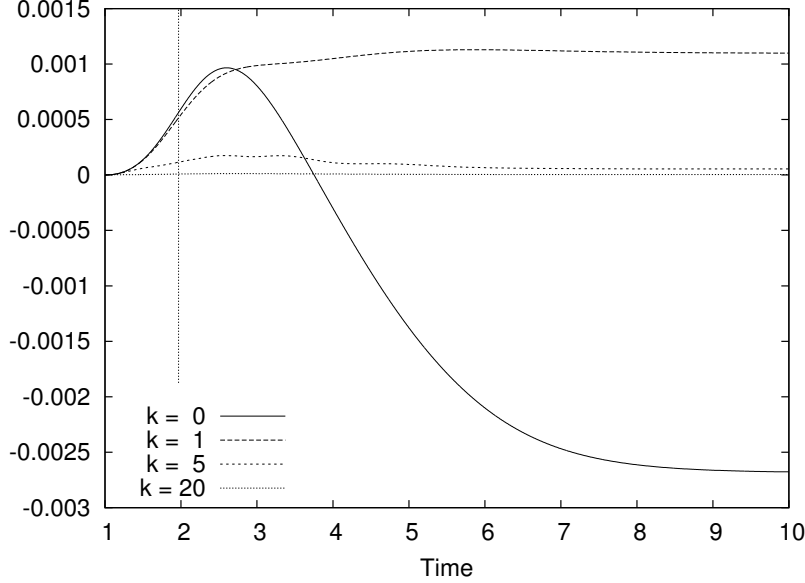


Figure 5. The growth of the density perturbations \mathcal{D}_{\parallel} in the dust background of section 6.1.2 for the wave numbers $k = k_{\parallel}/a_{10} = k_{\perp}/a_{20} = 0, 1, 5$ and 20. Initially, at $t_0 = 1$, $\tilde{\mathcal{T}} = \tilde{\mathcal{T}} = 0.001$. The time of the bounce is indicated with a dotted vertical line.

perturbations is relatively insensitive to wave number, we here only give them for $k = 1$ in figure 4. In the direction of anisotropy the density gradients first grow and reaches a maximum at approximately the time of the bounce and then decay into a small but nonzero value. For the perpendicular directions the behaviour is inversed. First a minimum is reached around the time of the bounce, and then the gradient grows towards a constant value. A comparison with the evolution of density perturbations in a bouncing closed and isotropic dust universe, see figure E3 in Appendix E, shows that the local minimum (maximum) is absent in the isotropic case.

Next, in picture 5, the growth of the density perturbations \mathcal{D}_{\parallel} for the case when there is an initial shear perturbation $\tilde{\mathcal{T}} = \tilde{\mathcal{T}} = 0.001$ (and the other perturbations are zero), is shown. In this case the density perturbations depend on the wave number k . The reason for this can be seen from the systems of equations (93)-(96) and (C.5)-(C.8), where $\hat{\Delta}(\tilde{\mathcal{T}} + \tilde{\mathcal{T}})$ and $\delta^2(\tilde{\mathcal{T}} + \tilde{\mathcal{T}})$ appear as source terms.

Note that all perturbations asymptotically approach constant values. This is consistent with that the background asymptotically approaches de Sitter. The result holds also for the cases 6.3.2 and 6.3.4, that both are dust solutions approaching de Sitter. That this result does not apply for the radiation solutions can be understood from that terms like $p/\mu = 1/3$ remain in the equations also in the limit $\mu \rightarrow 0$.

6.2. Recollapsing solutions

6.2.1. From ${}_+K_-$ to ${}_+K_-$, radiation Here the background, that is filled with radiation, starts close to an expanding Kasner, ${}_+K_-$, reaches a largest value, and

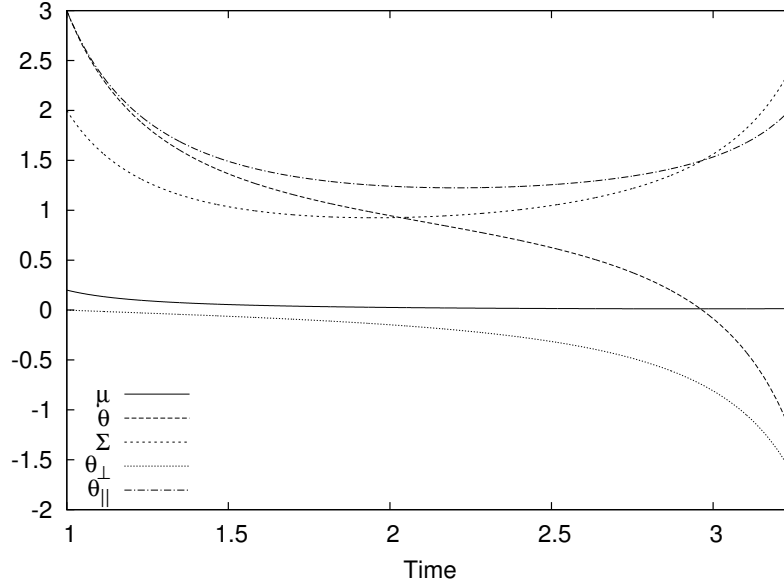


Figure 6. A radiation background that starts expanding, reaches a largest value and then recollapses. It starts close to $+K_-$ and ends at $-K_-$. Initial values at $t_0 = 1$ are given by $\mu_0 = 0.2$, $\theta_0 = 3$, $\Sigma_0 = 2$, $\theta_{\parallel} = 3$ and $\theta_{\perp} = 0$.

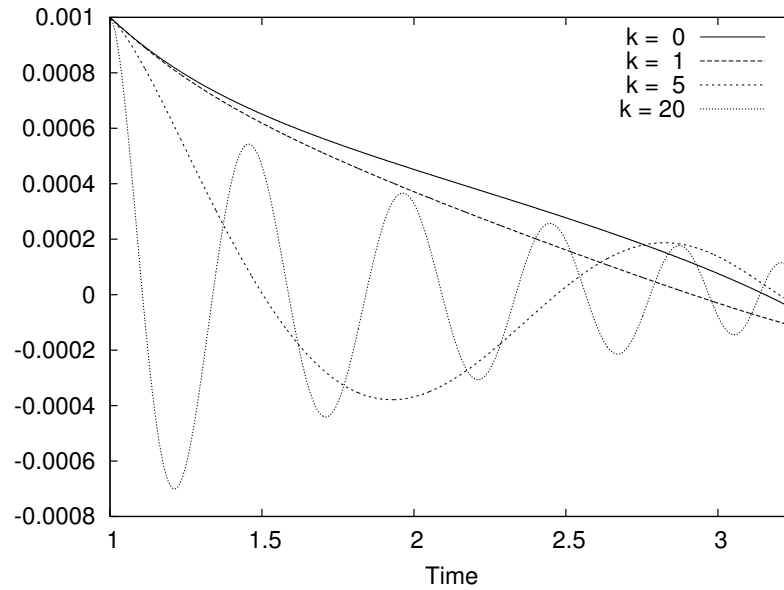


Figure 7. The growth of the density perturbations \mathcal{D}_{\parallel} in the background given by figure 6 for the wave numbers $k = k_{\parallel}/a_{10} = k_{\perp}/a_{20} = 0, 1, 5$ and 20. Initially $\dot{\mathcal{D}} = \mathcal{D} = 0.001$.

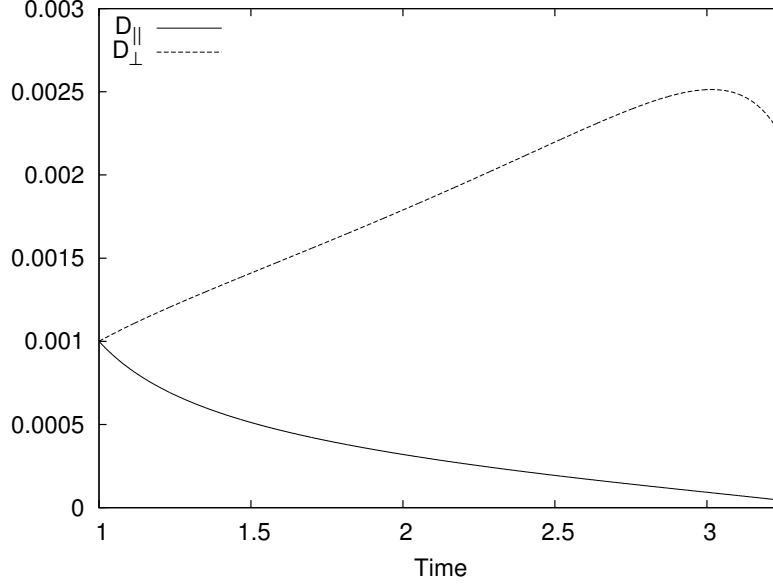


Figure 8. The growth of density perturbations \mathcal{D}_{\parallel} and \mathcal{D}_{\perp} in the dust background of section 6.2.2 for the wave numbers $k = k_{\parallel}/a_{10} = k_{\perp}/a_{20} = 1$. Initially $\hat{\mathcal{D}} = \mathcal{P} = 0.001$ and $\mu_0 = 0.2$, $\theta_0 = 3$, $\Sigma_0 = 2$, $\theta_{\parallel} = 3$ and $\theta_{\perp} = 0$.

then approaches a collapsing Kasner, $-K_-$. In the direction of anisotropy it is always expanding, whereas it is collapsing in the perpendicular directions. Initially this collapse velocity is negligible, but eventually it will dominate over the expansion in the direction of anisotropy. The evolutions of density, μ_0 , expansion, θ_0 , expansion in anisotropy direction, θ_{\parallel} , expansion in one of the perpendicular directions, θ_{\perp} and shear Σ_0 are depicted in figure 6. In figure 7 the growth of the density perturbations \mathcal{D}_{\parallel} for different values of the comoving wave numbers k_{\parallel} and k_{\perp} is shown. Initially $\hat{\mathcal{D}} = \mathcal{P} = 0.001$ and $\hat{\mathcal{Z}} = \mathcal{Z} = \hat{\mathcal{T}} = \tilde{\mathcal{T}} = \hat{\mathcal{S}} = \mathcal{S} = 0$.

As can be seen the density gradient in the direction of anisotropy decays for all k -numbers. In the collapsing directions the gradients grow.

6.2.2. From $+K_-$ to $-K_-$, dust This is a similar situation to the previous one, but with dust. The evolutions of the background quantities are once again close to those of the corresponding radiation case in 6.2.1, see figure 6. Also in this dust case the perturbations are rather insensitive to the wave number, and we hence only give them for one k -value in figure 8. As before the density perturbations decay in the expanding direction and initially grow in the contracting directions.

6.3. Solutions without a bounce

For all cases in this section the initial conditions of the perturbations are $\hat{\mathcal{D}} = \mathcal{P} = 0.001$ and $\hat{\mathcal{Z}} = \mathcal{Z} = \hat{\mathcal{T}} = \tilde{\mathcal{T}} = \hat{\mathcal{S}} = \mathcal{S} = 0$.

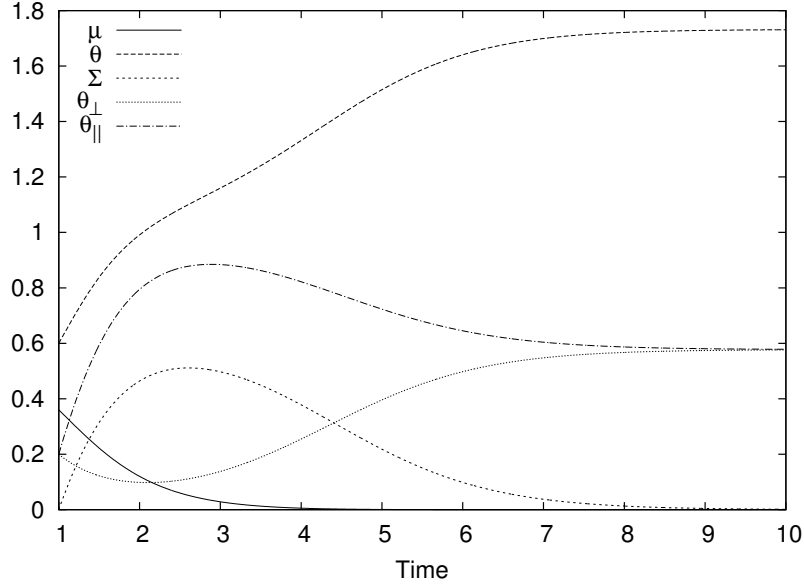


Figure 9. A radiation background passing close to expanding Friedmann, ${}_+F$, and ending at expanding de Sitter ${}_+dS$. Initial values at $t_0 = 1$ are given by $\mu_0 = 0.36$, $\theta_0 = 0.6$ and $\Sigma_0 = 0$.

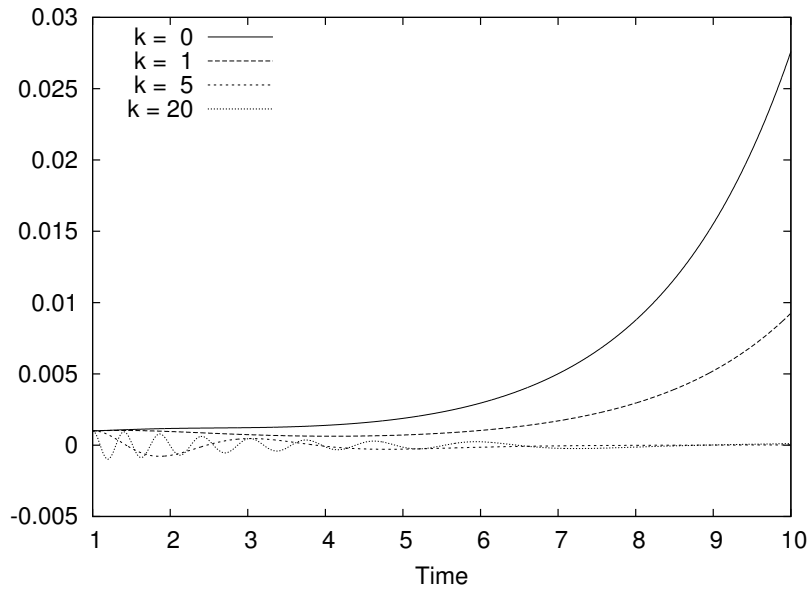


Figure 10. The growth of the density perturbations \mathcal{D}_\parallel in the background given by figure 9 for the wave numbers $k = k_\parallel/a_{10} = k_\perp/a_{20} = 0, 1, 5$ and 20. Initially $\dot{\mathcal{D}} = \mathcal{D} = 0.001$.

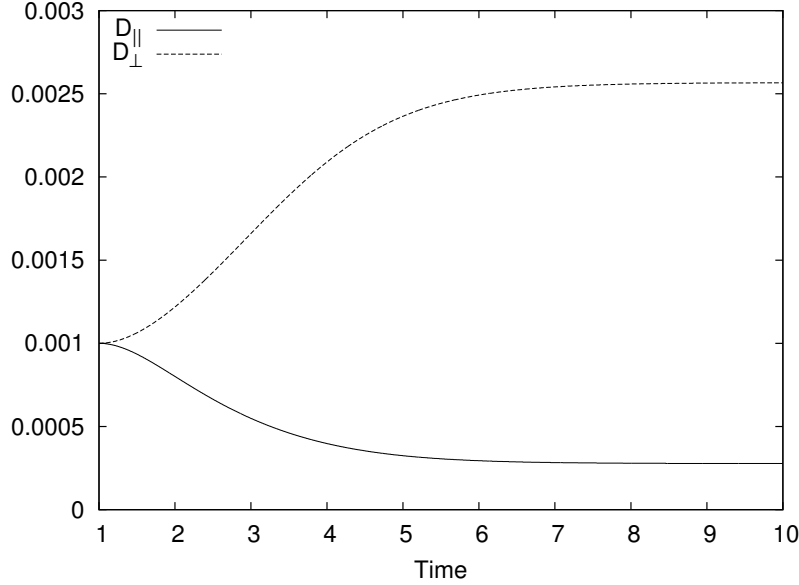


Figure 11. The growth of density perturbations \mathcal{D}_{\parallel} and \mathcal{D}_{\perp} in the dust background of section 6.3.2 for the wave numbers $k = k_{\parallel}/a_{10} = k_{\perp}/a_{20} = 1$. Initially $\hat{\mathcal{D}} = \mathcal{P} = 0.001$ and $\mu_0 = 0.36$, $\theta_0 = 0.6$ and $\Sigma_0 = 0$.

6.3.1. Friedmann to de Sitter, radiation This radiation background passes close to an expanding Friedmann, ${}_+F$, and ends at an expanding de Sitter, ${}_+dS$. It originates from an expanding Kasner, ${}_+K_+$, as can be seen by integrating backwards in time with the same initial conditions. Note also that at $t = 1$, the starting point in figure 9, the growth rate of the shear is nonzero. The evolutions of density, μ_0 , expansion, θ_0 , expansion in anisotropy direction, θ_{\parallel} , expansion in one of the perpendicular directions, θ_{\perp} and shear Σ_0 are depicted in figure 9. The expansion in the direction of anisotropy is for a period of time dominating over the expansion in the perpendicular directions. In figure 10 the growth of the density perturbations \mathcal{D}_{\parallel} for different values of the comoving wave numbers k_{\parallel} and k_{\perp} is shown. In the direction of anisotropy the super horizon modes $k = 0, 1$ eventually grows unboundedly, whereas for higher modes the amplitude slowly falls off. In the perpendicular directions the $k = 0, 1$ modes grow faster and the amplitudes of the higher modes slowly grow.

6.3.2. Friedmann to de Sitter, dust For the corresponding dust background to the radiation case of section 6.3.1, the evolutions of the background quantities are once again close to those of the radiation case, see figure 9. As in the previous dust models the perturbations are insensitive to the wave number. In figure 11 the growth of density perturbations for the comoving wave number $k = 1$ are shown. The density gradients in the direction of anisotropy falls off to a nonzero constant value, whereas in the perpendicular directions they grow towards a higher constant value.

6.3.3. ${}_+K_-$ to ${}_+dS$, radiation This radiation background starts close to an expanding Kasner, ${}_+K_-$, and ends at an expanding de Sitter, ${}_+dS$. The evolutions of

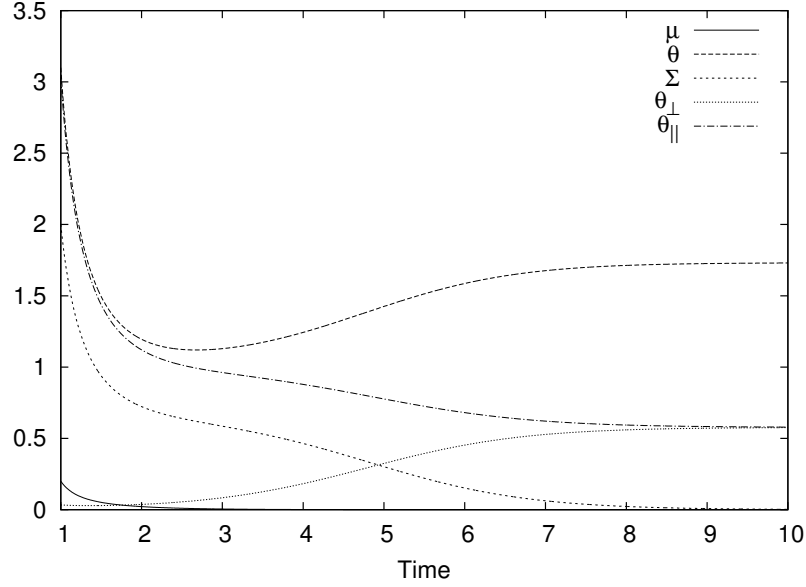


Figure 12. A radiation background starting close to expanding Kasner, ${}_+K_-$, and ending at expanding de Sitter ${}_+dS$. Initial values at $t_0 = 1$ are given by $\mu_0 = 0.2$, $\theta_0 = 3.1$ and $\Sigma = 2$.

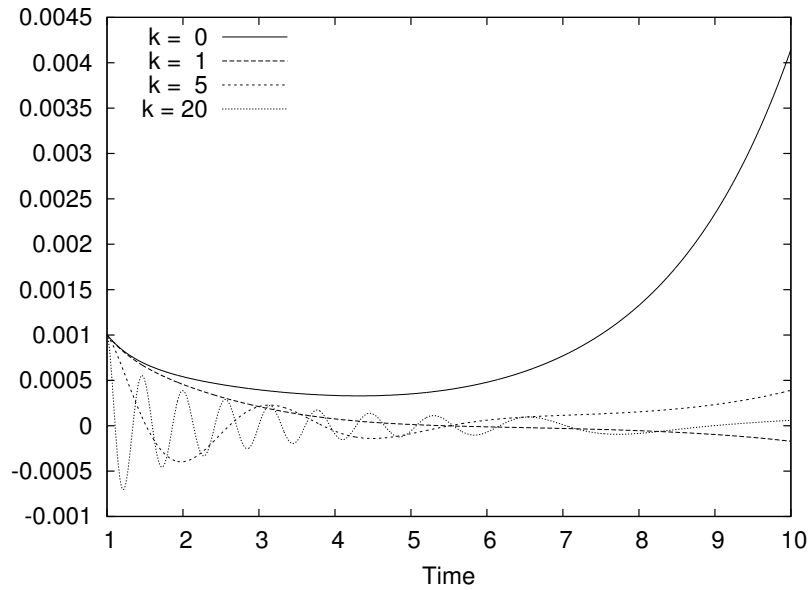


Figure 13. The growth of the density perturbations \mathcal{D}_\parallel in the background given by figure 12 for the wave numbers $k = k_\parallel/a_{10} = k_\perp/a_{20} = 0, 1, 5$ and 20. Initially $\dot{\mathcal{D}} = \mathcal{D} = 0.001$.

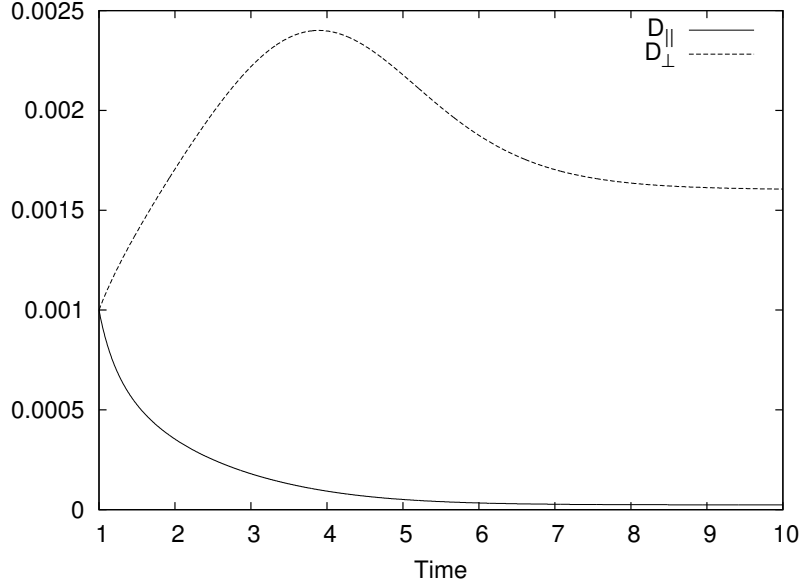


Figure 14. The growth of density perturbations \mathcal{D}_{\parallel} and \mathcal{D}_{\perp} in the dust background in section 6.3.4 for the wave numbers $k = k_{\parallel}/a_{10} = k_{\perp}/a_{20} = 1$. Initially $\hat{\mathcal{D}} = \mathcal{P} = 0.001$ and $\mu_0 = 0.2$, $\theta_0 = 3.1$ and $\Sigma = 2$.

density, μ_0 , expansion, θ_0 , expansion in anisotropy direction, θ_{\parallel} , expansion in one of the perpendicular directions, θ_{\perp} and shear Σ_0 are depicted in figure 12. In figure 13 the growth of the density perturbations \mathcal{D}_{\parallel} for different values of the comoving wave numbers k_{\parallel} and k_{\perp} is shown. In the direction of anisotropy the $k = 0$ mode initially decreases, but then turns and starts to grow unboundedly. The higher modes initially have decreasing amplitudes. Also these starts to increase, but more slowly, at about the same time as the $k = 0$ mode. In the perpendicular directions the modes do not show this initial decrease, but otherwise behave in a similar way.

6.3.4. $+K_-$ to $+dS$, dust This dust background starts close to an expanding Kasner, $+K_-$, and ends at an expanding de Sitter, $+dS$. For the evolutions of the background quantities see figure 12 for the corresponding radiation case, which shows a similar behaviour. In figure 14 the growth of density perturbations for $k = 1$ is given.

7. Summary

A closed system for scalar perturbations on Kantowski-Sachs cosmologies has been found in terms of gauge invariant variables. For long wavelengths and low densities some analytical results are obtained. By redefining some apparently singular terms, the system could be rewritten in a form suitable for numerical analysis. Due to the complexity of the governing equations, the choice of background, initial conditions and wave numbers, many different behaviors of the growth of the density perturbations can be obtained. In general the growth of density gradients is different in the direction of the anisotropy and in the perpendicular directions, which should influence the

formation of structures.

More importantly, growth rates are also affected by a bounce, see, e.g., figure 4. Typically the density gradients experience maxima (or minima) slightly after the bounce in the anisotropic Kantowski-Sachs models. A comparison with density growths in bouncing closed FLRW models, see Appendix E, shows that the amplitudes of the perturbations are roughly constant initially and that the local extrema around the time of the bounce are absent or very small in the isotropic case.

It was shown in [68] that a FLRW model with dust, radiation and cosmological constant cannot undergo a bounce, unless it took place after the formation of the quasars. In section 3.5 we use similar arguments in the Kantowski-Sachs case to show that a bounce in the directions perpendicular to the preferred direction cannot take place at all, whereas a bounce in the preferred direction cannot be excluded by these arguments. We also derive relations between present values of observational quantities including expansion and deceleration rates in the different directions. Since anisotropies in the Hubble and decelerations parameters cannot be excluded by present observations, see e.g. [70] and [71], it would be of interest to get better estimates of these parameters.

The present analysis can be extended to study tensor perturbations, including gravitational waves, and the coupling between scalar and tensor perturbations. In this way possible observational consequences of anisotropic bounces can be formulated in a way which makes it easy to compare with what happens in the standard case. The techniques developed here can also be used to develop a general framework for describing the evolution of cosmological perturbations in other non-standard cosmologies for example the Lemaître–Tolman–Bondi models [74]. Such studies are extremely important as they test the various rigidities which exist between different set of independent observables (from the background and at the level of perturbation theory) that can be used to test the underlying hypothesis of the standard cosmological model.

Acknowledgments

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Appendix A. Some approximate background-solutions with matter

Here some approximate solutions with $\mu \sim p \ll \Lambda$ to the system (53)-(55) are given.

With the bounce solution (49) as background the solution is given by

$$\mu_1 = C \cosh^{-\gamma}(\sqrt{\Lambda}t) \quad (\text{A.1})$$

$$\Sigma_1 - \frac{2}{3}\theta_1 = \cosh(\sqrt{\Lambda}t) \times \left[D + \gamma C \int \frac{dt}{\cosh^{\gamma+1}(\sqrt{\Lambda}t)} \right] \quad (\text{A.2})$$

$$\Sigma_1 + \frac{5}{6}\theta_1 = \cosh^{-2}(\sqrt{\Lambda}t) \times \left[E + \frac{(6-5\gamma)C}{4} \int \cosh^{2-\gamma}(\sqrt{\Lambda}t) dt \right] \quad (\text{A.3})$$

where C , D and E are constants of integration. The integrals in (A.2) and (A.3) can

be performed for dust, $\gamma = 1$, giving

$$\frac{C}{\sqrt{\Lambda}} \tanh(\sqrt{\Lambda}t) \quad \text{and} \quad \frac{C}{4\sqrt{\Lambda}} \sinh(\sqrt{\Lambda}t) \quad (\text{A.4})$$

respectively, and also for stiff matter, $\gamma = 2$, giving

$$\frac{2C}{\sqrt{\Lambda}} \left(\frac{\sinh(\sqrt{\Lambda}t)}{2 \cosh^2(\sqrt{\Lambda}t)} + \arctan(e^{\sqrt{\Lambda}t}) \right) \quad \text{and} \quad \frac{Ct}{4} \quad (\text{A.5})$$

respectively.

With the critical points $\pm X$, (45), as background, the solutions are given by

$$\mu_1 = Ce^{\mp\gamma\sqrt{\Lambda}t} \quad (\text{A.6})$$

$$\Sigma_1 - \frac{2}{3}\theta_1 = De^{\pm\sqrt{\Lambda}t} \mp \frac{\gamma C}{(\gamma+1)\sqrt{\Lambda}} e^{\mp\gamma\sqrt{\Lambda}t} \quad (\text{A.7})$$

$$\Sigma_1 + \frac{5}{6}\theta_1 = Ee^{\mp 2\sqrt{\Lambda}t} \pm \frac{(6-5\gamma)C}{4(2-\gamma)\sqrt{\Lambda}} e^{\mp\gamma\sqrt{\Lambda}t}. \quad (\text{A.8})$$

Note that since for all of these solutions some terms grow unboundedly, they are only valid for limited time intervals.

Appendix B. Calculation of some first order quantities

In the calculations several contractions of the type $x_{ab}y^{ab}$ of PSTF tensors

$$x_{ab} = X(n_a n_b - \frac{1}{2}N_{ab}) + 2X_{(a}n_{b)} + X_{ab} \quad (\text{B.1})$$

will appear. To first order contractions of two, three or four PSTF tensor are given by

$$x_{ab}y^{ab} = \frac{3}{2}XY, \quad x_{ab}y^{bc}z_c^a = \frac{3}{4}XYZ, \quad x_{ab}y^{bc}z_{cd}u^{da} = \frac{9}{8}XYZU \quad (\text{B.2})$$

respectively, so that, e.g., $\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab} = \frac{3}{4}\Sigma^2$.

In calculating the expression for $\dot{S} \equiv (\sigma^{ab}S_{ab})^\cdot$, needed for the equation for \dot{S}_a , the following term

$$\begin{aligned} \sigma^{ab}(\text{curl}H)_{ab} &= \sigma^{ab}(\text{curl}(\text{curl}\sigma))_{ab} \equiv \sigma^{ab}\eta^{cd}{}_{<a}\tilde{\nabla}_c(\text{curl}\sigma)_{b>d} = \\ \sigma^{ab}\eta^{cd}{}_a\tilde{\nabla}_c(\text{curl}\sigma)_{bd} &\equiv \sigma^{ab}\eta^{cd}{}_a\tilde{\nabla}_c\eta^{ef}{}_{<b}\tilde{\nabla}_e\sigma_{d>f} = \sigma^a{}_b\eta^c{}_{da}\eta^{ef}{}_{<b}\tilde{\nabla}_c\tilde{\nabla}_e\sigma^d{}_{>f}, \end{aligned} \quad (\text{B.3})$$

where it was used that σ_{ab} already is PSTF and that the projected derivative of η_{abc} is zero, will appear. Using the definition of PSTF and that

$$\eta_{abc}\eta^{def} = \begin{vmatrix} h_a^d & h_a^e & h_a^f \\ h_b^d & h_b^e & h_b^f \\ h_c^d & h_c^e & h_c^f \end{vmatrix} \quad (\text{B.4})$$

it simplifies to

$$\begin{aligned} \sigma^{ab}(\text{curl}H)_{ab} &= -\tilde{\nabla}_a\tilde{\nabla}^a(\sigma^2) + \frac{1}{2}\sigma^{ab}\tilde{\nabla}_b\tilde{\nabla}^c\sigma_{ca} + \sigma_{ab}\tilde{\nabla}_c\tilde{\nabla}^a\sigma^{bc} \\ &= -\tilde{\nabla}_a\tilde{\nabla}^a(\sigma^2) + \frac{1}{2}\sigma^{ab}\tilde{\nabla}_b\left(\frac{2}{3}\tilde{\nabla}_a\theta\right) + \sigma_{ab}\tilde{\nabla}_c\tilde{\nabla}^a\sigma^{bc}, \end{aligned} \quad (\text{B.5})$$

where (23) was used in the last step. With the definitions (71) and the 3-dimensional Ricci identity

$$\tilde{\nabla}_c\tilde{\nabla}_d\sigma_{ab} - \tilde{\nabla}_d\tilde{\nabla}_c\sigma_{ab} = {}^{(3)}R_{aecd}\sigma_b^e + {}^{(3)}R_{ebcd}\sigma_a^e, \quad (\text{B.6})$$

that holds with $\tilde{\nabla}_a$ since $\omega_{ab} = 0$, one gets

$$\sigma^{ab}(\text{curl}H)_{ab} = -\frac{1}{a}\tilde{\nabla}_a\mathcal{T}^a + \frac{1}{a}\sigma^{ab}\tilde{\nabla}_a\mathcal{Z}_b + \sigma^{ab}{}^{(3)}R_{acdb}\sigma^{cd} + {}^{(3)}R_{ab}\sigma^{bc}\sigma_c^a. \quad (\text{B.7})$$

The 3-Riemann tensor is given in terms of the projected 4-curvature, $R_{abcd\perp} \equiv h_a^e h_b^f h_c^g h_d^h R_{efgh}$, and the extrinsic curvature, $K_{ab} \equiv \tilde{\nabla}_{(a}u_{b)} = \theta/3 + \sigma_{ab}$, as

$${}^{(3)}R_{abcd} = R_{abcd\perp} - K_{ac}K_{bd} + K_{bc}K_{ad} \quad (\text{B.8})$$

and the 3-Ricci tensor by

$${}^{(3)}R_{ab} = S_{ab} + \frac{1}{3}h_{ab}{}^{(3)}R = S_{ab} + \frac{1}{3}h_{ab}\left(2\mu - \frac{2}{3}\theta + 2\sigma^2 + 2\Lambda\right). \quad (\text{B.9})$$

Then, on using that the 4-curvature can be decomposed as

$$R_{cd}^{ab} = \frac{2}{3}(\mu + 3p - 2\Lambda)u^{[a}u_{[c}h^{b]}_{d]} + \frac{2}{3}h^a_{[c}h^b_{d]} + 4u^{[a}u_{[c}E^{b]}_{d]} + 4h^{[a}_{[c}E^{b]}_{d]} + 2\eta^{abe}u_{[c}H_{d]e} + 2\eta_{cde}u^{[a}H^{b]e} \quad (\text{B.10})$$

(B.7) becomes

$$\sigma^{ab}(\text{curl}H)_{ab} = \frac{1}{a}\sigma^{ab}\tilde{\nabla}_a\mathcal{Z}_b - \frac{1}{a}\tilde{\nabla}_a\mathcal{T}^a + 2(\sigma^2)^2 - \frac{2}{3}\theta^2\sigma^2 + 3\tilde{S}\sigma^2 + 2\sigma^2(\mu + \Lambda) \quad (\text{B.11})$$

where \tilde{S} is defined from equation (74).

Appendix C. Perturbative equations in orthogonal directions

The perturbative equations, obtained by projecting (76)-(79) with N_{ab} , are

$$\dot{\mathcal{D}}_{\bar{a}} = (\mathcal{D}_{\bar{a}})^\cdot = \frac{\theta p}{\mu}\mathcal{D}_{\bar{a}} + \frac{1}{2}\Sigma\mathcal{D}_{\bar{a}} - \mathcal{Z}_{\bar{a}}(1 + \frac{p}{\mu}) \quad (\text{C.1})$$

$$\begin{aligned} \dot{\mathcal{Z}}_{\bar{a}} = (\mathcal{Z}_{\bar{a}})^\cdot = & -\frac{2}{3}\theta\mathcal{Z}_{\bar{a}} - \frac{1}{2}\mu\mathcal{D}_{\bar{a}} + \frac{1}{2}\Sigma\mathcal{Z}_{\bar{a}} - 2\mathcal{T}_{\bar{a}} + \frac{3}{2}\frac{\mu p'}{\mu + p}\left(\tilde{S} + \frac{3}{2}\Sigma^2\right)\mathcal{D}_{\bar{a}} \\ & - \frac{\mu p'}{\mu + p}\delta_a\tilde{\nabla}^b\mathcal{D}_b \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} \dot{\mathcal{T}}_{\bar{a}} = (\mathcal{T}_{\bar{a}})^\cdot = & -2\theta\mathcal{T}_{\bar{a}} + \frac{1}{2}\Sigma\mathcal{T}_{\bar{a}} - \mathcal{S}_{\bar{a}} - \frac{3}{2}\Sigma^2\mathcal{Z}_{\bar{a}} + \frac{3}{2}\Sigma\left(\tilde{S} + \theta\Sigma\right)\frac{\mu p'}{\mu + p}\mathcal{D}_{\bar{a}} - \\ & - \frac{\mu p'}{\mu + p}\Sigma\left(\frac{3}{2}\delta_a n^c\tilde{\nabla}_c\mathcal{D} - \frac{1}{2}\delta_a\tilde{\nabla}^b\mathcal{D}_b\right) \end{aligned} \quad (\text{C.3})$$

$$\begin{aligned} \dot{\mathcal{S}}_{\bar{a}} = (\mathcal{S}_{\bar{a}})^\cdot = & \left(\Sigma^2 + 2\frac{\tilde{S}^2}{\Sigma^2}\right)\mathcal{T}_{\bar{a}} + \left(\frac{5}{2}\Sigma - \frac{5}{3}\theta - 2\frac{\tilde{S}}{\Sigma}\right)\mathcal{S}_{\bar{a}} + \frac{p'\mu}{\mu + p}\tilde{S}\left(\frac{5}{2}\theta\Sigma + \frac{3}{2}\tilde{S} - \frac{3}{2}\Sigma^2\right)\mathcal{D}_{\bar{a}} \\ & + \mu\Sigma^2\mathcal{D}_{\bar{a}} - \Sigma\left(\frac{5}{2}\tilde{S} + \frac{2}{3}\Sigma\theta\right)\mathcal{Z}_{\bar{a}} + \frac{p'\mu}{\mu + p}\left[\frac{1}{2}\left(\tilde{S} - \frac{1}{3}\theta\Sigma + 2\Sigma^2\right)\delta_a\tilde{\nabla}^b\mathcal{D}_b - \right. \\ & \left. \frac{3}{2}\left(\tilde{S} - \frac{1}{3}\theta\Sigma + \Sigma^2\right)\delta_a n^b\tilde{\nabla}_b\mathcal{D}\right] + \frac{3}{2}\Sigma\delta_a n^b\tilde{\nabla}_b\mathcal{Z} - \frac{1}{2}\Sigma\delta_a\tilde{\nabla}^b\mathcal{Z}_b - \delta_a\tilde{\nabla}^b\mathcal{T}_b. \end{aligned} \quad (\text{C.4})$$

The system obtained by taking the 2-divergence of (C.1)-(C.4) is

$$\dot{\mathcal{P}} = \left[\theta\left(\frac{p}{\mu} - \frac{1}{3}\right) + \Sigma\right]\mathcal{P} - \left(1 + \frac{p}{\mu}\right)\mathcal{Z} \quad (\text{C.5})$$

$$\begin{aligned} \dot{\mathcal{Z}} = & (\Sigma - \theta)\mathcal{Z} + \left[-\frac{1}{2}\mu + \frac{3}{2}\frac{\mu p'}{\mu + p}\left(\tilde{S} + \frac{3}{2}\Sigma^2\right)\right]\mathcal{P} - 2\Sigma^2\tilde{\mathcal{T}} - 2\frac{\Sigma}{\tilde{S}}\mathcal{S} \\ & - \frac{\mu p'}{\mu + p}\delta^2\left[\hat{\mathcal{D}} + \mathcal{P}\right] \end{aligned} \quad (\text{C.6})$$

$$\begin{aligned} \dot{\tilde{\mathcal{T}}} = & - \left(\frac{1}{3}\theta - \Sigma + \frac{\Sigma^3}{\tilde{S}} \right) \tilde{\mathcal{T}} - \left(\frac{\Sigma^2}{\tilde{S}^2} + \frac{1}{\tilde{S}} \right) \mathcal{S} - \left[\frac{\Sigma\mu}{\tilde{S}} + \frac{\mu p'}{\mu+p} \left(\theta - \frac{3}{2}\Sigma \right) \right] \mathcal{P} + \left(1 + \frac{2}{3} \frac{\Sigma\theta}{\tilde{S}} \right) \mathcal{Z} + \\ & \frac{\mu p'}{\mu+p} \frac{1}{\tilde{S}} \left[\left(\frac{1}{2}\Sigma - \frac{1}{3}\theta \right) \delta^2 \hat{\mathcal{D}} - \left(\Sigma - \frac{1}{6}\theta \right) \delta^2 (\mathcal{P}) \right] - \frac{1}{\tilde{S}} \delta^2 (\hat{\mathcal{Z}} - \frac{1}{2}\mathcal{Z}) + \frac{\Sigma}{\tilde{S}} \delta^2 (\tilde{\mathcal{T}} + \hat{\mathcal{T}}) + \\ & \frac{1}{\tilde{S}^2} \delta^2 (\hat{\mathcal{S}} + \mathcal{S}) \end{aligned} \quad (\text{C.7})$$

$$\begin{aligned} \dot{\mathcal{S}} = & \left[\mu\Sigma^2 + \frac{\mu p'}{\mu+p} \tilde{S} \left(\frac{5}{2}\theta\Sigma + \frac{3}{2}\tilde{S} - \frac{3}{2}\Sigma^2 \right) \right] \mathcal{P} - \left(\frac{2}{3}\theta\Sigma + \frac{5}{2}\tilde{S} \right) \Sigma\mathcal{Z} + \left(\Sigma^4 + 2\tilde{S}^2 \right) \tilde{\mathcal{T}} + \\ & \left(\frac{\Sigma^3}{\tilde{S}} - 2\theta + 3\Sigma \right) \mathcal{S} + \Sigma\delta^2 \left(\hat{\mathcal{Z}} - \frac{1}{2}\mathcal{Z} \right) - \Sigma^2\delta^2 (\tilde{\mathcal{T}} + \hat{\mathcal{T}}) - \frac{\Sigma}{\tilde{S}} \delta^2 (\hat{\mathcal{S}} + \mathcal{S}) + \\ & \frac{\mu p'}{\mu+p} \left[\left(-\tilde{S} + \frac{1}{3}\theta\Sigma - \frac{1}{2}\Sigma^2 \right) \delta^2 \hat{\mathcal{D}} + \frac{1}{2} \left(\tilde{S} - \frac{1}{3}\theta\Sigma + 2\Sigma^2 \right) \delta^2 (\mathcal{P}) \right]. \end{aligned} \quad (\text{C.8})$$

Appendix D. Perturbations around analytical solutions

Appendix D.1. Perturbations around ${}_+X$

With the ${}_+X$ -solution as background the system (93)-(96) for the hat-variables reduces to

$$\dot{\hat{\mathcal{D}}} = -\frac{5}{3}\sqrt{\Lambda}\hat{\mathcal{D}} - \hat{\mathcal{Z}}, \quad \dot{\hat{\mathcal{Z}}} = -\frac{7}{3}\sqrt{\Lambda}\hat{\mathcal{Z}} - 2\hat{\mathcal{T}} \quad (\text{D.1})$$

$$\dot{\hat{\mathcal{T}}} = -\frac{11}{3}\sqrt{\Lambda}\hat{\mathcal{T}} - \hat{\mathcal{S}} - \frac{2}{3}\Lambda\hat{\mathcal{Z}}, \quad \dot{\hat{\mathcal{S}}} = \frac{22}{27}\Lambda^{3/2}\hat{\mathcal{Z}} + \frac{22}{9}\Lambda\hat{\mathcal{T}} \quad (\text{D.2})$$

if the harmonic numbers are put to zero (corresponding to large wave-lengths of the perturbations). From these equations one finds the fourth order equation (109) for the density perturbations $\hat{\mathcal{D}}$ with solution (110). The other quantities are then obtained as

$$\hat{\mathcal{Z}} = -\dot{\hat{\mathcal{D}}} - \frac{5}{3}\sqrt{\Lambda}\hat{\mathcal{D}}, \quad \hat{\mathcal{T}} = \frac{1}{2}\ddot{\hat{\mathcal{D}}} + 2\sqrt{\Lambda}\dot{\hat{\mathcal{D}}} + \frac{35}{18}\Lambda\hat{\mathcal{D}} \quad (\text{D.3})$$

$$\hat{\mathcal{S}} = -\frac{1}{2}\hat{\mathcal{D}}^{(3)} - \frac{23}{6}\sqrt{\Lambda}\ddot{\hat{\mathcal{D}}} - \frac{155}{18}\Lambda\dot{\hat{\mathcal{D}}} - \frac{325}{54}\Lambda^{3/2}\hat{\mathcal{D}} \quad (\text{D.4})$$

giving

$$\hat{\mathcal{D}} = A_1 + A_2 + A_4, \quad \hat{\mathcal{Z}} = \sqrt{\Lambda}(2A_1 - A_2) - A_3 \quad (\text{D.5})$$

$$\hat{\mathcal{T}} = \frac{\Lambda}{3}(4A_1 + \frac{5}{2}A_2) + \frac{\sqrt{\Lambda}}{3}A_3, \quad \hat{\mathcal{S}} = -\frac{\Lambda^{3/2}}{3}(4A_1 + \frac{11}{2}A_2) \quad (\text{D.6})$$

as initial conditions at $t = 0$.

Appendix D.2. Perturbations around vacuum bounce solution

With the vacuum bounce solution (49) as background the system (93)-(96) for the hat-variables reduces to

$$\dot{\hat{\mathcal{D}}} = -\frac{5}{3}\theta\hat{\mathcal{D}} - \hat{\mathcal{Z}} \quad (\text{D.7})$$

$$\dot{\hat{\mathcal{Z}}} = -\frac{7}{3}\theta\hat{\mathcal{Z}} - \frac{8}{9}\theta^2\hat{\mathcal{T}} + 2\frac{\theta}{\Lambda}\hat{\mathcal{S}} \quad (\text{D.8})$$

$$\dot{\tilde{\mathcal{T}}} = -\frac{1}{3}\theta \left(5 - \frac{4}{3}\frac{\theta^2}{\Lambda}\right) \tilde{\mathcal{T}} + \frac{3}{2\Lambda} \left(1 - \frac{2}{3}\frac{\theta^2}{\Lambda}\right) \hat{\mathcal{S}} + \left(1 - \frac{2}{3}\frac{\theta^2}{\Lambda}\right) \hat{\mathcal{Z}} \quad (\text{D.9})$$

$$\dot{\hat{\mathcal{S}}} = \frac{2}{9}\theta \left(5\Lambda - \frac{4}{3}\theta^2\right) \hat{\mathcal{Z}} + \frac{8}{9} \left(\Lambda^2 + \frac{2}{9}\theta^4\right) \tilde{\mathcal{T}} - 2\theta \left(1 + \frac{2}{9}\frac{\theta^2}{\Lambda}\right) \hat{\mathcal{S}} \quad (\text{D.10})$$

(once again only considering the long wave-length limit). It is suitable to change the independent variable to θ through

$$\dot{\psi} = \frac{d\psi}{d\theta} \dot{\theta} = \frac{d\psi}{d\theta} (\Lambda - \theta^2).$$

The equation (112) for $\hat{\mathcal{D}}$ with solution (113) is then obtained. The other quantities are given by

$$\hat{\mathcal{Z}} = -\frac{5}{3}\theta \hat{\mathcal{D}} - (\Lambda - \theta^2) \frac{d\hat{\mathcal{D}}}{d\theta} \quad (\text{D.11})$$

$$\begin{aligned} \tilde{\mathcal{T}} = & \frac{1}{8\theta^2\Lambda} \left(\frac{5}{3} (9\Lambda^2 - 45\Lambda\theta^2 - 8\theta^4) \hat{\mathcal{D}} - 5\theta (3\Lambda + 4\theta^2) (\Lambda - \theta^2) \frac{d\hat{\mathcal{D}}}{d\theta} + \right. \\ & \left. 3 (3\Lambda - 5\theta^2) (\Lambda - \theta^2) \frac{d^2\hat{\mathcal{D}}}{d\theta^2} - 9\theta (\Lambda - \theta^2)^3 \frac{d^3\hat{\mathcal{D}}}{d\theta^3} \right) \end{aligned} \quad (\text{D.12})$$

$$\begin{aligned} \hat{\mathcal{S}} = & -\frac{5}{54} \hat{\mathcal{D}} \theta (57\Lambda + 8\theta^2) - \frac{1}{18} \frac{d\hat{\mathcal{D}}}{d\theta} (33\Lambda + 20\theta^2) (\Lambda - \theta^2) - \\ & \frac{5}{6} \frac{d^2\hat{\mathcal{D}}}{d\theta^2} \theta (\Lambda - \theta^2)^2 - \frac{1}{2} \frac{d^3\hat{\mathcal{D}}}{d\theta^3} (\Lambda - \theta^2)^3. \end{aligned} \quad (\text{D.13})$$

Taylor expanding $\hat{\mathcal{D}}$ to second order around $\theta = 0$ gives

$$\hat{\mathcal{D}} = \tilde{A}_1 + (\tilde{A}_2 + \tilde{A}_3)\theta - \frac{5}{6} \frac{\tilde{A}_1}{\Lambda} \theta^2 + \dots \quad (\text{D.14})$$

$$\hat{\mathcal{Z}} = -(\tilde{A}_2 + \tilde{A}_3)\Lambda + \left(\frac{11}{6}\tilde{A}_2 + \frac{1}{3}\tilde{A}_3\right)\theta^2 + \dots \quad (\text{D.15})$$

$$\tilde{\mathcal{T}} = -\frac{27}{8}\tilde{A}_4 - \frac{9}{4}\tilde{A}_3\theta + \frac{9}{16} \frac{\tilde{A}_4}{\Lambda} \theta^2 + \dots \quad (\text{D.16})$$

$$\hat{\mathcal{S}} = \left(\frac{2}{3}\tilde{A}_2 - \frac{5}{6}\tilde{A}_3\right)\Lambda^2 - 3\tilde{A}_4\Lambda\theta - \left(\frac{11}{9}\tilde{A}_2 + \frac{13}{18}\tilde{A}_3\right)\Lambda\theta^2 + \dots, \quad (\text{D.17})$$

where $\tilde{A}_1 = A_1\Lambda^{5/6}$, $\tilde{A}_2 = A_2\Lambda^{5/6}$, $\tilde{A}_3 = A_3\Lambda^{1/3}$ and $\tilde{A}_4 = A_4\Lambda^{5/6}$.

Appendix E. Perturbations on bouncing closed Friedmann

Closed Friedmann models with a positive cosmological constant may undergo a bounce [24]. The evolution equations are given by

$$\dot{\mu} = -\theta(\mu + p) \quad (\text{E.1})$$

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \frac{1}{2}(\mu + 3p - 2\Lambda). \quad (\text{E.2})$$

With an equation of state $p = (\gamma - 1)\mu$ the following integral

$$\theta^2 = 3\mu + 3\Lambda - 3C\mu^{2/3\gamma} \quad (\text{E.3})$$

is found. Here closed models correspond to $C > 0$. In figure E1 a bouncing radiation solution is shown. It starts from a contracting state, goes through a bounce and then

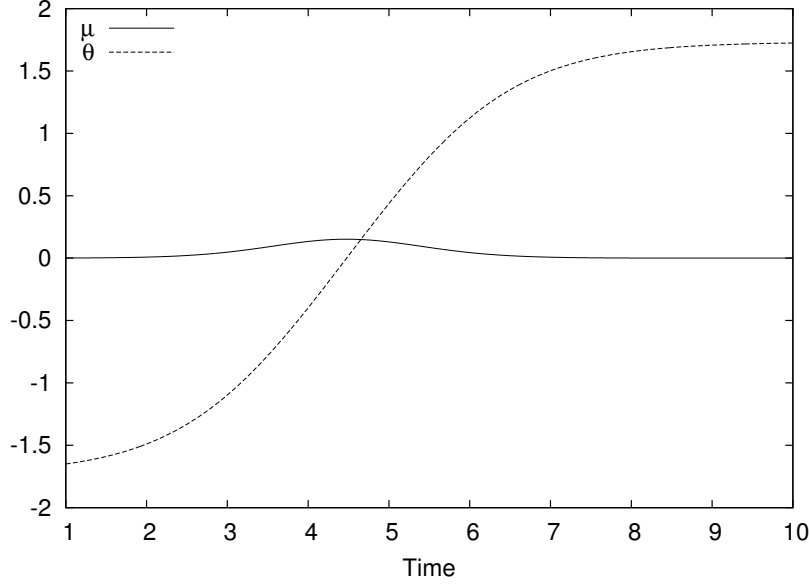


Figure E1. A closed initially contracting Friedmann model with radiation, that goes through a bounce and then starts expanding. Initial values at $t_0 = 1$ are given by $\mu_0 = 0.001$, $\theta_0 = -1.65$. $\Lambda = 1$ and $C = 2.96$.

asymptotically approaches de Sitter.

To study the growth of density perturbations it is sufficient to use the density gradient $\mathcal{D}_a = \frac{a\tilde{\nabla}_a\mu}{\mu}$ and expansion gradient $\mathcal{Z}_a = a\tilde{\nabla}_a\theta$, in which the system now closes [57]:

$$\dot{\mathcal{D}}_a = \frac{\theta p}{\mu} \mathcal{D}_a - \mathcal{Z}_a \left(1 + \frac{p}{\mu}\right) \quad (\text{E.4})$$

$$\dot{\mathcal{Z}}_a = -\frac{2}{3}\theta\mathcal{Z}_a - \frac{1}{2}\mu\mathcal{D}_a + \frac{\mu p'}{\mu + p} \left(-\mu - \Lambda + \frac{1}{3}\theta^2\right) \mathcal{D}_a - \frac{\mu p'}{\mu + p} \tilde{\nabla}_a \tilde{\nabla}^b \mathcal{D}_b. \quad (\text{E.5})$$

Here $p' \equiv dp/d\mu$. In accordance with equations (90) and (99) we define the scalar density gradient as

$$\mathcal{D} = a\tilde{\nabla}^a \mathcal{D}_a. \quad (\text{E.6})$$

A harmonic decomposition

$$\mathcal{D} = \sum_{\tilde{k}} \mathcal{D}_{\tilde{k}} Q_{\tilde{k}} \quad \text{where} \quad \tilde{\nabla}^2 Q_{\tilde{k}} = -\frac{\tilde{k}^2}{a^2} Q_{\tilde{k}} \quad \text{and} \quad \dot{Q}_{\tilde{k}} = 0 \quad (\text{E.7})$$

is then done. Here $\tilde{k}^2 = n(n+2)$ for $n = 1, 2, \dots$, [72, 73].

The growth of the density perturbations for the radiation background in picture E1 is shown for different wave-numbers in picture E2. Initially the amplitudes of the perturbations are roughly constant with a small dip in the amplitude of the $k = 0$ mode, that serves as an approximation for the long wavelength limit, before the bounce.

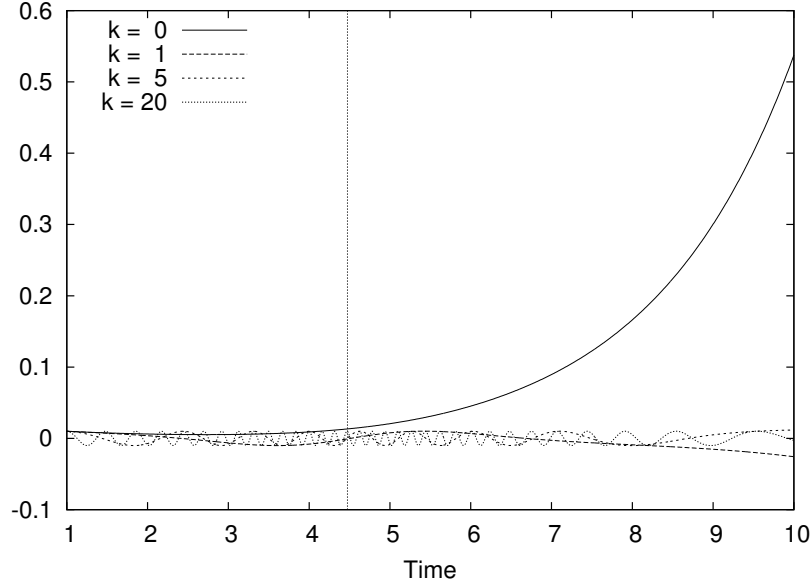


Figure E2. The growth of the density perturbations \mathcal{D} in the background given by figure E1 for the wave numbers $k = k/a_0 = 0, 1, 5$ and 20 . Initially, at $t_0 = 1$, $\mathcal{D} = 0.01$. The time of the bounce is indicated with a dotted vertical line.

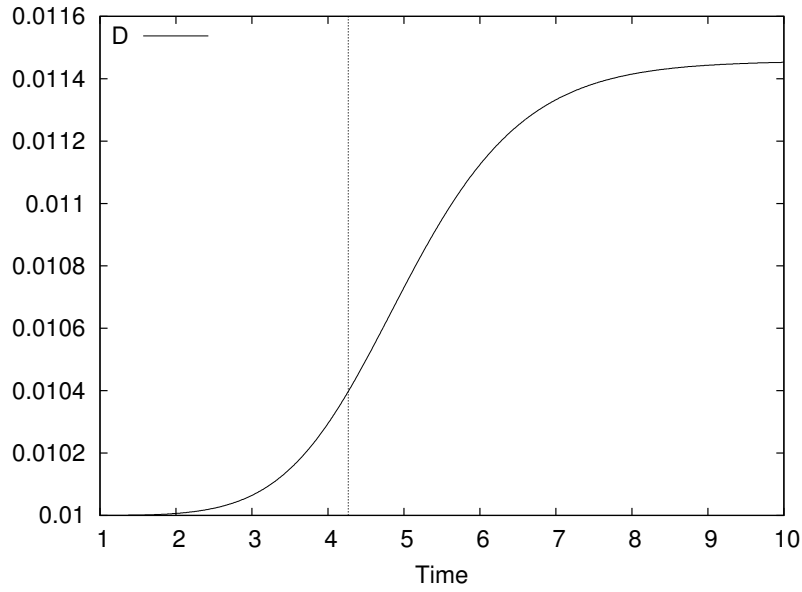


Figure E3. The growth of the density perturbations \mathcal{D} for a dust background. Initially, at $t_0 = 1$, $\mathcal{D} = 0.01$. Initial conditions for background $\mu_0 = 0.001$, $\theta_0 = -1.65$. $\Lambda = 1$ and $C = 9.35$. The time of the bounce is indicated with a dotted vertical line.

In the corresponding dust case the behaviour of the background quantities are similar to the radiation case, whereas the evolution of the perturbations is independent of the wave-number, as can be seen from (E.5). See E3 for the growth of \mathcal{D} .

Bouncing closed Friedmann universes containing dust and radiation with cosmological constant are excluded by observations, due to an argument by Börner and Ehlers [68]. We here repeat their argument for the dust case in terms of our variables. Using the definition of the scale factor a in terms of the expansion, $\theta \equiv 3\dot{a}/a$, and Eq. (E.1) one obtains $\mu = \mu_0(a_0/a)^3$, where the subscript 0 refers to present values. The constant in (E.3) is given by $3C\mu_0^{2/3} = 3\mu_0 + 3\Lambda - \theta_0^2$ in terms of present values. At the bounce we have $\dot{a}_* = 0$ and $\ddot{a}_* > 0$. Using this into Eqs. (E.2) and (E.3) one then obtains the following inequality

$$\Omega_m [(z+1)^3 - 3(z+1) + 2] < 2 \quad (\text{E.8})$$

where $\Omega_m \equiv 3\mu_0/\theta_0^2$ is the present fraction of dust relative to the critical density and $z = a_0/a_* - 1$ is the redshift corresponding to the time of the bounce. If the bounce took place before the formation of quasars, that are observed at redshifts $z > 4$, Ω_m cannot overcome the value 0.02, in contradiction with observations.

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